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On Some Solutions of the 3D Navier-Stokes Equations

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Abstract

In this paper, there are determined various solutions of the stationary Navier-Stokes equations, which describe the 3D flow of an incompressible liquid or gas, i.e., solutions containing the components of the velocity of flow – the functions u, v, w and the created pressure P. We mention that in the paper a series of exact solutions are obtained, in which the viscosity coefficient λ participates explicitly.

Keywords: stationary three-dimensional Navier-Stokes equations, exact solutions, method of separation of variables, viscosity, pressure, velocity of flow of a liquid or gas.
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1 Introduction

In the present paper, the Navier-Stokes equations are studied in the stationary three-dimensional case. In this case, the Navier-Stokes equations represent a system containing four differential equations contained in system (1) and equation (2). Until today, the examined problem has not been definitively solved. The complexity of the problem lies in the fact that the equations in the system (1) are non-linear. A method is not developed that would allow us to determine all the solutions of this system. Determining the solutions of the system of Navier-Stokes equations is an important mathematical problem and has various applications in fluid and gas mechanics.

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The following systems of partial differential equations is examined:

$$\begin{cases} \dfrac{P_x}{\mu} + uu_x + vu_y + wu_z = \lambda\Delta u + F_x, \\ \dfrac{P_y}{\mu} + uv_x + vv_y + wv_z = \lambda\Delta v + F_y, \\ \dfrac{P_z}{\mu} + uw_x + vw_y + ww_z = \lambda\Delta u + F_z. \end{cases} \tag{1}$$

$$u_x + v_y + w_z = 0. \tag{2}$$

Here $P = P(x, y, z)$, $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$, $F = F(x, y, z)$; $u_x = \dfrac{\partial u}{\partial x}$; $\Delta u = u_{xx} + u_{yy} + u_{zz}$; $x, y, z \in R$.

Systems (1), (2) describe the process of the stationary flow of a liquid or gas in three-dimensional space. The system consisting of (1), (2) represents the Navier-Stokes equations in the case of the three-dimensional stationary motion of a viscous incompressible fluid or gas.

The function P represents the pressure of the liquid, and the functions u, v, w represent the flow velocity component of the liquid or gas. The function F represents the external force and is potential in nature. The constants $\lambda > 0$ and $\mu > 0$ are a determined parameter of the studied liquid's or gas viscosity and density. We mention here that $\lambda = c/R_e, c > 0$, where R_e is the Reynolds number.

Regarding the derivation of the equations of system (1), (2) and the meaning of the physical processes described by this system, consult the works [1], [2]. A number of solutions to the equations in the two-dimensional case have been determined in the papers [3], [4].

Regarding the application of various methods for determining the solutions of linear and nonlinear equations with partial derivatives, you can see the works [6].

2 Determination of solutions that do not explicitly contain the viscosity coefficient

The following theorem is true:

Theorem 1. *Let, in the connected domain D, the functions $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$, and $P(x,y,z)$ admit continuous partial derivatives up to and including the second order. If in this domain the functions u, v, w, and P verify the following equalities:*

$$\begin{aligned} u_y &= v_x, u_z = w_x, w_y = v_z, u_x + v_y + w_z = 0, \\ P &= \mu\left[F + C - 0,5\left(u^2 + v^2 + w^2\right)\right], \end{aligned} \tag{3}$$

then these functions are the solutions of systems (1), (2) for any constant C.

Proof. Adding to both sides of the first equation of system (1) the expression $0,5\left(v^2+w^2\right)_x$, in the second equation the expression $0,5\left(u^2+w^2\right)_y$ and in the third equation of this system the expression $0,5\left(u^2+v^2\right)_z$, we obtain the following system, which is equivalent to system (1):

$$\begin{cases} G_x = \lambda\Delta u - v\left(u_y - v_x\right) + w\left(w_x - u_z\right), \\ G_y = \lambda\Delta v + u\left(u_y - v_x\right) - w\left(v_z - w_y\right), \\ G_z = \lambda\Delta w - u\left(w_x - u_z\right) + v\left(v_z - w_y\right), \end{cases} \tag{4}$$

where $G = \mu^{-1}P - F + 0,5\left(u^2+v^2+w^2\right)$.

If in the connected domain D the functions u, v, w, and P admit continuous partial derivatives up to and including the second order and $u_y = v_x$, $u_z = w_x$, $w_y = v_z$, $u_x + v_y + w_z = 0$, then $\Delta u = \Delta v = \Delta w = 0$. Therefore, from the system (4), it follows that in this domain, the function G is a constant and the pressure is equal $P = \mu\left[F + C - 0,5\left(u^2+v^2+w^2\right)\right]$.

Theorem 1 is proved.

Below we present an example of solutions to the system (1), (2) that verify the conditions of Theorem 1.

Example 1 For the flow velocity components, we have $u = cxt^{-1,5}$, $v = cyt^{-1,5}$, $w = czt^{-1,5}$; $t = x^2 + y^2 + z^2$, c is arbitrary constant. The pressure P is determined from (3).

The following theorem generates a series of solutions of equations (1), (2) which can be obtained from formulas (3).

Theorem 2. *Let $f = a(x)b(y)c(z)$, where the functions $a(x)$, $b(y)$, and $c(z)$ are doubly differentiable and satisfy the following equations:*

$a^{''} = ra, b^{''} = sb, c^{''} = lc; r + s + l = 0$, *here r, s, l are constants.*

Then for the flow velocity components of the problem (1), (2) we have $u = f_x, v = f_y, w = f_z$. The pressure P is determined from (3).

Proof. Under the conditions of Theorem 2, the function f has continuous partial derivatives up to the second order inclusive. Because $u = f_x$, $v = f_y$, $w = f_z$, then the following equalities $u_y = v_x$, $u_z = w_x$, $w_y = v_z$ are valid. From condition (2), we obtain that

$$a^{''}(x)b(y)c(z) + a(x)b^{''}(y)c(z) + a(x)b(y)c^{''}(z) = 0.$$

Dividing both sides of this equation by $a(x)b(y)c(z)$, we get the following equation: $\frac{a^{''}(x)}{a(x)} + \frac{b^{''}(y)}{b(y)} + \frac{c^{''}(z)}{c(z)} = 0 \Leftrightarrow a^{''} = ra$, $b^{''} = sb$, $c^{''} = lc$; $r + s + l = 0$; $r, s, l - const$. Thus, the functions u, v, w are determined. After this, it is determined the pressure P from (3).

Theorem 2 is proved.

Suppose that under the conditions of Theorem 2: $r > 0$, $l > 0$, $s = -(r + l) < 0$. Then from the equations $a^{''} = ra$, $b^{''} = sb$, $c^{''} = lc$, we will obtain:

$$a(x) = a_1 e^{\sqrt{r}x} + a_2 e^{-\sqrt{r}x}; b(y) = b_1 e^{\sqrt{l}y} + b_2 e^{-\sqrt{l}y},$$

$$c(z) = c_1 \cos\left(\sqrt{r + l}z\right) + c_2 \sin\left(\sqrt{r + l}z\right),$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ – arbitrary constants.

Then

$$f = \left(a_1 e^{\sqrt{r}x} + a_2 e^{-\sqrt{r}x}\right) \times \left(b_1 e^{\sqrt{l}y} + b_2 e^{-\sqrt{l}y}\right) \times$$
$$\times \left(c_1 \cos\sqrt{r + l}z + c_2 \sin\sqrt{r + l}z\right).$$

Analogously, we obtain f in the other cases with compliance with the condition $r + s + l = 0$.

Example 2 The components of velocity are: $u = 3c\sin(5z)e^{3x+4y}$, $v = 4c\sin(5z)e^{3x+4y}$, $w = 5c\cos(5z)e^{3x+4y}$, c is constant.

The pressure P is determined from (3).

3 The solutions that explicitly contain the viscosity coefficient λ

The following two theorems give solutions in which the viscosity parameter λ participates explicitly

Theorem 3. *Let $u = f_1(t)$, $v = f_2(t)$, $w = f_3(t)$, $t = x + y + z$, where the functions $f_k, k = 1, 2, 3$ are differentiable up to and including the third order. Then*

$$u = \frac{-1}{3C}\left[(C_1 + C_3)\,t + C_2 + C_4 + 3\lambda C^{-1}(C_1 + C_3) + (C_0 + C_5)\,e^{\frac{Ct}{3\lambda}}\right],$$
$$v = \frac{1}{3C}\left[(2C_1 - C_3)\,t + 2C_2 - C_4 + 3\lambda C^{-1}(2C_1 - C_3) + \right.$$
$$\left. + (2C_0 - C_5)\,e^{\frac{Ct}{3\lambda}}\right],$$
$$w = C + \frac{1}{3C}\left[(2C_3 - C_1)\,t + 2C_4 - C_2 + 3\lambda C^{-1}(2C_3 - C_1) + \right.$$
$$\left. + (2C_5 - C_0)\,e^{\frac{Ct}{3\lambda}}\right].$$

where $C, C_0, C_1, C_2, C_3, C_4, C_5$ are constant.

For pressure we get:

$$P = \mu\left[F - 3\lambda\left(u'(t) + v'(t)\right) + uv + 0,5\left(u^2 + v^2 - w^2\right)\right]. \qquad (5)$$

Proof. Let $u = f_1(t)$, $v = f_2(t)$, $w = f_3(t)$, $t = x + y + z$. From (2), we immediately deduce that $f_3' = -f_1' - f_2' \Rightarrow f_3 = C - f_1 - f_2$; C is some constant.

Then from system (4), we get:

$$\begin{cases} G_x = 3\lambda f_1'' - f_2(f_1' - f_2') + (C - f_1 - f_2)(-2f_1' - f_2') = \\ 3\lambda f_1'' - C(2f_1' + f_2') + Q, \\ G_y = 3\lambda f_2'' + f_1(f_1' - f_2') - (C - f_1 - f_2)(2f_2' + f_1') = \\ 3\lambda f_2'' - C(2f_2' + f_1') + Q, \\ G_z = -3\lambda(f_1'' + f_2'') + f_1(2f_1' + f_2') + f_2(2f_2' + f_1') = \\ -3\lambda(f_1'' + f_2'') + Q. \end{cases} \tag{6}$$

Here $Q = (f_1 f_2)' + 2(f_1 f_1') + 2(f_2 f_2')$.

Since the functions $f_k, k = 1, 2, 3$ are differentiable up to and including the third order, system (6) will have solutions in any connected domain if and only if the following conditions $G_{xy} = G_{yx}$, $G_{xz} = G_{zx}$, $G_{yz} = G_{zy}$ are met. Applying these conditions, we arrive at a system in which only the functions f_k participate:

$$\begin{cases} 3\lambda(f_1'' - f_2'')' - C(f_1' - f_2')' = 0, \\ 3\lambda(2f_1'' + f_2'')' - C(2f_1' + f_2')' = 0, \\ 3\lambda(f_1'' + 2f_2'')' - C(f_1' + 2f_2')' = 0. \end{cases}$$

We notice that by subtracting from the second equation the third equation of the given system, we will obtain exactly its first equation. Therefore, we omit the last equation of the system, and we integrate the first two equations twice or consecutively. Thus, we arrive at a system equivalent to system (6):

$$\begin{cases} 3\lambda\,(f_1 - f_2)' - C\,(f_1 - f_2) = C_1 t + C_2, \\ 3\lambda\,(2f_1 + f_2)' - C\,(2f_1 + f_2) = C_3 t + C_4, \end{cases} \tag{7}$$

where $t = x + y + z$, C, C_1, C_2, C_3, C_4 are arbitrary constants.

Solving system (7), we will obtain:

$$\begin{cases} f_1 - f_2 = -C^{-1}(C_1 t + C_2 + 3\lambda C^{-1} C_1 + C_0 e^{\frac{C_t}{3\lambda}}), \\ 2f_1 + f_2 = -C^{-1}(C_3 t + C_4 + 3\lambda C^{-1} C_3 + C_5 e^{\frac{C_t}{3\lambda}}). \end{cases} \tag{8}$$

Solving system (8) by Cramer's method, we will obtain:

$$f_1 = \frac{-1}{3C}\left[(C_1+C_3)\,t + C_2 + C_4 + 3\lambda C^{-1}\,(C_1+C_3) + \right.$$
$$\left. + (C_0+C_5)\,e^{\frac{C_t}{3\lambda}}\right],$$
$$f_2 = \frac{1}{3C}\left[(2C_1-C_3)\,t + 2C_2 - C_4 + 3\lambda C^{-1}\,(2C_1-C_3) + \right.$$
$$\left. + (2C_0-C_5)\,e^{\frac{C_t}{3\lambda}}\right].$$

Here $u = f_1(t)$, $v = f_2(t)$, $w = f_3(t) = C - f_1(t) - f_2(t)$.

The pressure P is determined from system (6). Theorem 3 is proved.

Example 3 The components of velocity are: $u = t + 3\lambda C^{-1} + e^{\frac{C_t}{3\lambda}} + c$, $v = -t - 3\lambda C^{-1} - 3e^{\frac{C_t}{3\lambda}} - 2c$, $w = C + 2e^{\frac{C_t}{3\lambda}} + c$; C, c are constant.

The pressure P is determined from system (5).

Theorem 4. *Let $u = C_1 f(t)$, $v = C_2 f(t)$, $w = C_3 f(t)$; $t = ax+by+cz$, where a, b, c, C_1, C_2, C_3 are constants and function $f(t)$ admits continuous derivatives up to and including the third order. If the following conditions $aC_1 + bC_2 + cC_3 = 0$, $bC_1 \neq aC_2$, $aC_3 \neq cC_1$ and $cC_2 \neq bC_3$ are valid, then $f(t) = \dfrac{kt^2+mt+h}{\lambda}$, where k, m, h are arbitrary constant.*

Theorem 4 is proved analogously to Theorem 3.

4 Conclusions

In the current article, there are determined a lot of exact solutions of the stationary three-dimensional Navier-Stokes equations. We mention, that the solutions determined by Theorems 1 and 2 presented in example (1), (2) do not explicitly depend on the viscosity.

In the other solutions obtained in Theorems 3 and 4, the dependence of the solutions on the viscosity parameter is explicitly indicated.

The values of this constants can be determined based on initial conditions and boundary conditions of the physical problems examinates.

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