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# List of Normal 3-Isohedral Spherical Tilings for Group Series $2 * n$

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**Abstract**

Tilings of the 2-dimensional sphere with disks which fall into 3 transitivity classes under the group action are studied. For isometry group series $2 * n$, $n = 1, 2, \ldots$, we list all normal by Grünbaum and Shephard 3-isohedral spherical tilings.

**Keywords:** 3-isohedral tiling, sphere, normal tiling, isometry group series, Delone class.

We begin with reminding basic concepts. Let $W$ be a tilng of the 2-dimensional sphere with topological disks and $G$ be a discrete isometry group of the sphere.

**Definition 1.** *The tiling $W$ is called $k$-isohedral with respect to the group $G$ if $G$ maps $W$ onto itself and the tiles of $W$ fall into exactly $k$ transitivity classes under the group $G$.*

**Definition 2.** *Consider all possible pairs $(W, G)$ where the tiling $W$ of the sphere is $k$-isohedral with respect to the group $G$. Two pairs $(W, G)$ and $(W', G')$ are said to belong to the same Delone class if there exists a homeomorphism $\varphi$ of the sphere which maps the tiling $W$ onto the tiling $W'$ and for isometry groups the relation $G = \varphi^{-1}G'\varphi$ holds.*

In a tiling, a connected component of the intersection of two and more different disks is defined to be a vertex of the tiling if it is a single point or to be an edge of the tiling otherwise. A Delone class is fundamental if the group $G$ acts simply transitively on the set of tiles or is non-fundamental otherwise.

For spherical groups we use the Conway's orbifold symbol, characterizing the orbifold which is the quotient of the sphere by the group.

There are 7 countable series and 7 sporadic discrete isometry groups of the sphere.

A tiling of the sphere is called normal by B. Grünbaum and G. C. Shephard [1] if it satisfies the following conditions:

SN1. Each tile is a topological disk.

SN2. The intersection of any set of tiles is a connected (possibly empty) set.

SN3. Each edge of the tiling has two endpoints which are vertices of the tiling.

Both fundamental and non-fundamental isohedral (1-isohedral) tilings of the sphere with disks are described in [1].

In [2] the author developed some general methods for obtaining $k$-isohedral ($k \geq 2$) tilings of the Euclidean plane, the sphere and the hyperbolic plane. In particular, the splitting procedure can be applied to fundamental $(k-1)$-isohedral tilngs with disks yielding fundamental $k$-isohedral tilings with disks. By this way the enumeration of Delone classes of fundamental 2-isohedral tilings of the sphere with disks was obtained [3]. Also the same methods in terms of Delaney–Dress symbols were implemented in algorithms and computer programs [4, 5].

Recently the author has applied the splitting procedure to all 29 series of Delone classes of fundamental 2-isohedral tilings of the sphere with disks for group series $2 * n$, $n = 1, 2, ...$. It has resulted in 499 series of Delone classes of fundamental 3-isohedral spherical tilings with disks. The number 499 coincides with the numerical results given in [5].

We select from the obtained 3-isohedral tilings those which satisfy the above normality conditions SN1–SN3. Below we list the obtained Delone classes of 3-isohedral tilings by indicating, for each Delone class, 3 cycles of vertex valencies of tiles.

For group series $2 * n$, $n = 1, 2, ...$, that corresponds to series of 3-dimensional point isometry groups $\widetilde{2N} \cdot m = \widetilde{2N} : 2$, $N = 1, 2, ...$, there are 64 series of Delone classes of normal 3-isohedral tilings of the sphere: $[3^4; 3^3.4^2; 3^2.4.2n.4]$, $[3^4; 3^2.4^2; 3^3.4.2n.4]$, $[3^4; 3^2.4.4n; 3^3.4.4n]$, $[3^2.4^2; 3.4.2n.4; 3^3.4^2]$, $[3^3.6; 3^2.4.6; 3.4.2n.6]$, $[3.4^3; 3.4^3; 3.4.2n.4]$,

$[3^3; 3^2.4.2n.4; 3^4.4^2]$, $[3^2.4; 3^2.4.2n.4; 3.4.3.4^2]$, $[3.4^2; 3^3.4^2; 3^2.4.2n.4]$, $[3^3; 3^2.4^2; 3^4.4.2n.4]$, $[3^3; 3^2.4.4n; 3^4.4.4n]$, $[3^2.4; 3^2.4^2; 3.4.3.4.2n.4]$, $[3.4^2; 3^24^2; 3^3.4.2n.4]$, $[3^2.4; 3^2.4.4n; 3.4.3.4.4n]$, $[3^2.6; 3.4.2n.6; 3^3.4.6]$, $[3.4^2; 3^2.4.4n; 3^3.4.4n]$, 2 different series $[3.4^2; 3.4^3; 3.4^2.2n.4]$, $[3.4^2; 3.4.2n.4; 3.4^4]$, $[3.4.6; 3^3.6; 3^2.4.2n.6]$, $[3.4.4n; 3^2.4^2; 3^3.4.4n]$, $[3.4.4n; 3^2.4.4n; 3^3.4^2]$, $[4^2.2n; 3^2.4^2; 3.4^4]$, $[3.4^2; 3.4^2.4n; 3.4^2.4n]$, $[3.4^2; 3.4.6^2; 3.4.2n.6]$, 2 different series $[3.4^2; 3.4.4n.4; 3.4.4n.4]$, $[3.4.6; 3.4.6.4; 3.4.2n.6]$, $[3.4.6; 3.4.6^2; 3.4.2n.4]$, $[3.4.4n; 3.4^3; 3.4^2.4n]$, $[3.4.6n; 3^3.6n; 3^2.4.6n]$, $[3.6.4n; 3^3.6; 3^2.6.4n]$, $[4^3; 4^4; 4^3.2n]$, $[4^2.2n; 4^4; 4^4]$, $[4.6.2n; 3^2.4.6; 3.4^2.6]$, $[3^2.6; 3.4.6; 3^3.4.2n.6]$, $[3.4^2; 4^2.2n; 3^2.4^2]$, $[3^2.6; 3.6.4n; 3^3.6.4n]$, $[3^2.6n; 3.4.6n; 3^3.4.6n]$, $[3.4.6; 3.4.6; 3.4.6.2n.4]$, $[3.4.6; 4.6.2n; 3^2.4^2.6]$, $[3.4.4n; 3.4.4n; 3.4^4]$, $[4^3; 4^3; 4^4.2n]$, $[4^3; 4^2.2n; 4^5]$, $[3.4.6; 3.6.4n; 3.4.6.4n]$, $[3.4.8; 3.8^2; 3.4.2n.8]$, $[3.4.4n; 3.6.4n; 3.4.6^2]$, $[3.4.6n; 3.4.6n; 3.4.6n.4]$, 2 different series $[3.6^2; 6^2.2n; 3^2.6^2]$, $[4^3; 4^2.4n; 4^3.4n]$, $[4^2.6; 4^2.6; 4^2.6.2n]$, $[4^2.6; 4^2.6; 4.6.4.2n]$, 2 different series $[4^2.6; 4.6.2n; 4^3.6]$, $[4^2.2n; 4.6^2; 4^3.6]$, $[4^2.4n; 4^2.4n; 4^4]$, $[3.8^2; 3.8.4n; 3.8.4n]$, 2 different series $[4^2.6; 4.6.4n; 4.6.4n]$, $[4^2.8; 4.8^2; 4.8.2n]$, $[4^2.6n; 4^2.6n; 4^2.6n]$, $[4.6^2; 4.6^2; 6^2.2n]$ and $[4.6^2; 4.6.2n; 6^3]$, $n = 1, 2, ...$.

Remark that for isometry group series $*nn$, $nn$, $*22n$ and $n*$, $n = 1$, 2, ..., the Delone classes of normal 3-isohedral tilings of the sphere were listed in [6].

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