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On graded 2-absorbing and strongly 2-absorbing secondary submodules

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Abstract. In this paper, we introduce the concepts of graded 2-absorbing and strongly 2-absorbing secondary submodules. A number of results and basic properties of these graded submodule classes are presented.

1. Introduction

Throughout this article, we assume that A is a commutative $\mathfrak{G}$-graded ring with identity and D is a unitary graded A-module.

The concept of graded secondary submodules was introduced and discussed in [9]. A graded primary ideal concept was introduced and discussed by Refai and Al-Zoubi in [15]. In [5], the concept of graded 2-absorbing ideals was introduced and studied by Al-Zoubi, Abu-Dawwas and Çeken as a generalization of graded prime ideals. The notion of graded 2-absorbing ideals was extended to graded 2-absorbing submodules by Al-Zoubi and Abu-Dawwas in [6] and studied in [3]. The dual notion of graded 2-absorbing submodules (that is, graded 2-absorbing (resp. graded strongly 2-absorbing) second submodules) was presented and studied in [1, 14]. In [7, 16], the concept of graded 2-absorbing primary ideals was presented and studied. As a generalization of graded 2-absorbing primary ideals, the authors introduced and studied graded 2-absorbing primary submodules in [12, 2].

1.1. Objectives and Motivation

This paper aims to introduce the concepts of graded 2-absorbing and graded strongly 2-absorbing secondary submodules as the dual notions of graded 2-absorbing primary submodules and present some related results.

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The motivation for introducing these graded 2-absorbing secondary submodules lies in the fundamental importance of duality in module theory, which often leads to the discovery of new, complementary structures and properties. By studying this dual class, we aim to capture phenomena related to the "comultiplication" nature of modules, which are often overlooked when focusing solely on primary-type conditions. The dualization provides a richer, more symmetric framework, allowing for the comprehensive analysis of graded module decompositions and the behavior of graded prime ideals $P = Ann_A(U)$ associated with graded second submodules U.

1.2. Conceptual Distinction and Duality Principle

The gr-2^{abs}-secondary submodule is a natural generalization within the framework of module theory duality. Conceptually, the gr-2^{abs}-secondary submodule differs from the classical graded secondary submodule (gr-secondary) primarily in the number of factors involved in the absorption condition.

While gr-secondary submodules are defined by an absorption property concerning a single factor $r \in h(A)$, the gr-2^{abs}-secondary condition requires an absorption property involving the product of two homogeneous elements, $r_g t_h \in h(A)$. This generalization is guided by the fundamental Primary/Secondary Duality principle, which is a form of lattice-theoretic duality applied to graded modules. In this duality, the condition on the Graded Radical ($Grad(K)$) used for primary submodules is dualized to a condition on the Graded Second Radical ($GSec(C)$) for secondary submodules. Specifically, while graded 2-absorbing primary submodules are defined via annihilator containment (in terms of ideals of the ring), the dual concept is intrinsically linked to the notion of the graded second radical $GSec(C)$ and the structure of completely graded irreducible submodules.

2. Preliminaries

In this section we will give the definitions and results which are required in the next section.

Definition 2.1. (a) Let $\mathfrak{S}$ be a group with identity e and A be a commutative ring with identity 1_A. Then A is called a $\mathfrak{S}$-graded ring if there exist additive subgroups A_g of A indexed by the elements $g \in \mathfrak{S}$ such that $A = \bigoplus_{g\in\mathfrak{S}} A_g$ and $A_g A_h \subseteq A_{gh}$ for all $g, h \in \mathfrak{S}$. The elements of

A_g are called homogeneous of degree g. The set of all homogeneous elements of A is denoted by $h(A)$, i.e. $h(A) = \bigcup_{g\in\mathfrak{S}} A_g$, see [13].

(b) Let $A = \bigoplus_{g\in\mathfrak{S}} A_g$ be $\mathfrak{S}$-graded ring. An ideal P of A is called a graded ideal if $P = \sum_{h\in\mathfrak{S}} P\cap A_h = \sum_{h\in\mathfrak{S}} P_h$. By $P \leq^{id}_{\mathfrak{S}} A$, we mean that P is a $\mathfrak{S}$-graded ideal of A. Also, by $P <^{id}_{\mathfrak{S}} A$, we mean that P is a proper $\mathfrak{S}$-graded ideal of A, see [13].

(c) A left A-module D is said to be a $\mathfrak{S}$-graded A-module if $D = \bigoplus_{g\in\mathfrak{S}} D_g$ with $A_gD_h \subseteq D_{gh}$ for all $g, h \in \mathfrak{S}$, where D_g is an additive subgroup of D for all $g \in \mathfrak{S}$. The elements of D_g are called homogeneous of degree g. The set of all homogeneous elements of D is denoted by $h(D)$, i.e, $h(D) = \bigcup_{g\in\mathfrak{S}} D_g$. Note that D_h is an A_e-module for every $h \in \mathfrak{S}$, see [13].

(d) A submodule U of D is called a graded submodule of D if $U = \oplus_{h\in\mathfrak{S}}(U\cap D_h) := \oplus_{h\in\mathfrak{S}} U_h$. By $U \leq^{sub}_{\mathfrak{S}} D$, we mean that U is a $\mathfrak{S}$-graded submodule of D and by $\{0\} \neq U \leq^{sub}_{\mathfrak{S}} D$, we mean that U is a non-zero a $\mathfrak{S}$-graded submodule of D. Also, by $U <^{sub}_{\mathfrak{S}} D$, we mean that U is a proper $\mathfrak{S}$-graded submodule of D, see [13].

(e) If $U \leq^{sub}_{\mathfrak{S}} D$, then $(U :_A D) = \{a \in A \mid aD \subseteq U\} \leq^{id}_{\mathfrak{S}} A$, see [8]. Furthermore, the annihilator of U in A is denoted and defined by $Ann_A(U) = \{a \in A \mid aU = \{0\}\}$.

(f) $K <^{id}_{\mathfrak{S}} A$ is called a graded prime (briefly, gr-prime) ideal if $ab \in K$ where $a, b \in h(A)$, then $a \in K$ or $b \in K$. The graded radical of a graded ideal K, denoted by $Grad(K)$, is the set of all $t = \sum_{h\in\mathfrak{S}} t_h \in A$ such that for each $h \in \mathfrak{S}$ there exists $n_h \in \mathbb{N}$ with $t_h^{n_h} \in K$. Note that, if r is a homogeneous element, then $r \in Grad(K)$ if and only if $r^n \in K$ for some $n \in \mathbb{N}$, see [15].

(g) $K <^{id}_{\mathfrak{S}} A$ is called a *graded primary* (briefly, *gr-primary*) *ideal* if $r_gs_h \in K$ where $r_g, s_h \in h(A)$, then either $r_g \in K$ or $s_h \in Grad(K)$, see [15].

(h) $U <^{id}_{\mathfrak{S}} D$ is called a graded prime (briefly, gr-prime) submodule if $am \in U$ where $a \in h(A)$ and $m \in h(D)$, , then either $a \in (U :_A D)$ or $m \in U$, see [8].

(i) $\{0\} \neq U \leq^{sub}_{\mathfrak{S}} D$ is called graded second (briefly, gr-second) if for each $a \in h(A)$, the graded A-homomorphism $\Psi : U \to U$ defined by

$\Psi(x) = ax$ is either surjective or zero. In other words, U is a graded second submodule of D if $aU = U$ or $aU = 0$ for every $a \in h(A)$. Then $P = Ann_A(U)$ is a graded prime ideal of A and U is called a P-graded second submodule. The graded second spectrum of D, denoted by $GSpec^s(D)$, is the set of all graded second submodules of D, see [11].

(j) $U <^{su}_{\mathfrak{G}} D$ is called a completely graded irreducible if $U = \cap_{\alpha\in\Lambda} U_\alpha$, where $\{U_\alpha\}_{\alpha\in\Lambda}$ is a family of graded submodule of D, then $U = U_\alpha$ for some $\alpha \in \Lambda$. Every proper graded submodule of D is the intersection of all completely graded irreducible submodules containing it, see [1].

(k) A graded A-module D is called graded comultiplication (briefly, gr-com) module if for every $U \leq^{sub}_{\mathfrak{G}} D$, there exists a $P \leq^{id}_{\mathfrak{G}} A$ with $U = (0 :_D P)$, equivalently, for each $U \leq^{sub}_{\mathfrak{G}} D$, we have $U = (0 :_D Ann_A(U))$, see [4, 10].

Definition 2.2. (a) $\{0\} \neq U \leq^{sub}_{\mathfrak{G}} D$ is called a graded 2-absorbing second (briefly, gr-2^{abs}-second) submodule of D if whenever r, $t \in h(A)$, V is a completely graded irreducible submodule of D, and $rtU \subseteq V$, then $rU \subseteq V$ or $tU \subseteq V$ or $rt \in Ann_A(U)$, see [1, 14].

(b) $\{0\} \neq U \leq^{sub}_{\mathfrak{G}} D$ is called a graded strongly 2-absorbing second (briefly, gr-2^{abs}_{st}-second) submodule of D if whenever r, $t \in h(A)$, V_1, V_2 are completely graded irreducible submodules of D, and $rtU \subseteq V_1 \cap V_2$, then $rU \subseteq V_1 \cap V_2$ or $tU \subseteq V_1 \cap V_2$ or $rt \in Ann_A(U)$, see [1, 14].

(c) $K <^{id}_{\mathfrak{G}} A$ is said to be a graded 2-absorbing primary (briefly, gr-2^{abs}-primary) ideal of A if whenever $r_g, s_h, t_i \in h(D)$ with $r_g s_h t_i \in K$, then $r_g s_h \in K$ or $r_g t_i \in Grad(K)$ or $s_h t_i \in Grad(K)$, see [7].

(d) $\{0\} \neq U \leq^{sub}_{\mathfrak{G}} D$ is called graded secondary (briefly, gr-secondary) if for every homogeneous element of A either divides U or is nilpotent on U. In other words, U is a graded secondary submodule of D if $rU = U$ or $r^n U = 0$ for some integer n, for every $r \in h(A)$, see [9].

(e) For a graded A-submodule C of D, the graded second radical of C is defined as the sum of all graded second A-submodules of D contained in C, and its denoted by $GSec(C)$. If C does not contain any graded second A-submodule, then $GSec(C) = \{0\}$. The graded second spectrum of D is the collection of all graded second A-submodules, and it is represented by the symbol $GSpec^s(D)$, see [14]. On the other hand, the

set of all graded prime A-submodules of D is called the graded spectrum of D, and is denoted by $GSpec(D)$. The map $\phi : GSpec^s(D) \to GSpec(A/Ann_A(D))$ defined by $\phi(C) = Ann_A(C)/Ann_A(D)$ is called the natural map of $GSpec^s(D)$, see [8].

Remark 2.3. Let U and H are two graded submodule of an graded A-module. To prove $U \subseteq H$, its enough to show that if J is a completely graded irreducible submodule of D with $H \subseteq J$, then $U \subseteq J$, see [14, Remark 1.2].

3. Results

Definition 3.1. A nonzero graded submodule C of D is called a graded 2-absorbing secondary (briefly, gr-2^{abs}-secondary) submodule of D if whenever $r_g, t_h \in h(A)$, V is a completely graded irreducible submodule of D, and $r_g t_h\, C \subseteq V$, then $r_g\, GSec(C) \subseteq V$ or $t_h\, GSec(C) \subseteq V$ or $r_g t_h \in Ann_A(C)$. By a gr-2^{abs}-secondary module, we mean a graded module which is gr-2^{abs}-secondary submodule of itself.

Example 3.2. Let $A = \mathbb{Z}$ and $\mathfrak{S} = \mathbb{Z}_2$. Then A is a $\mathfrak{S}$-graded ring with $A_0 = \mathbb{Z}$ and $A_1 = \{0\}$. Consider $D = \mathbb{Z}$ as a $\mathbb{Z}$-module, D is a $\mathfrak{S}$-graded module with $D_0 = \mathbb{Z}$ and $D_1 = \{0\}$. Every graded submodule of D is not gr-secondary. But $GSec(\mathbb{Z}) = 0$, so we have every graded submodule of D is gr-2^{abs}-secondary.

Lemma 3.3. *Let $I = \bigoplus_{g\in\mathfrak{S}} I_g \leq^{id}_{\mathfrak{S}} A$ and $C \leq^{sub}_{\mathfrak{S}} D$. If C is gr-2^{abs}-secondary submodule of D and whenever $t_h \in h(A)$, V is completely graded irreducible submodule of D and $g \in \mathfrak{S}$ with $I_g t_h\, C \subseteq V$, then $I_g\, GSec(C) \subseteq V$ or $t_h\, GSec(C) \subseteq V$ or $I_g t_h \subseteq Ann_A(C)$.*

Proof. Assume that $t_h GSec(C) \nsubseteq V$ and $I_g t_h \nsubseteq Ann_A(C)$. Then there exists $x_g \in I_g$ such that $x_g t_h \notin Ann_A(C)$. Since C is gr-2^{abs}-secondary submodule of D and $x_g t_h C \subseteq V$, we have $x_g\, GSec(C) \subseteq V$. We show that $I_g\, GSec(C) \subseteq V$. Let $y_g \in I_g$, then $(x_g + y_g)t_h C \subseteq V$, and so either $(x_g + y_g)\, GSec(C) \subseteq V$ or $(x_g+y_g)t_h \subseteq Ann_A(C)$. If $(x_g+y_g)\, GSec(C) \subseteq V$, then since $x_g\, GSec(C) \subseteq V$ we have $y_g\, GSec(C) \subseteq V$. If $(x_g+y_g)t_h \subseteq Ann_A(C)$, then $y_g t_h \notin Ann_A(C)$. Thus $y_g t_h C \subseteq V$, implies that $y_g\, GSec(C) \subseteq V$, so $I_g\, GSec(C) \subseteq V$. □

Lemma 3.4. *Let $I = \bigoplus_{g\in\mathfrak{S}} I_g \leq^{id}_{\mathfrak{S}} A$ and $J = \bigoplus_{g\in\mathfrak{S}} J_g \leq^{id}_{\mathfrak{S}} A$. If C is gr-2^{abs}-secondary submodule of D and whenever V is completely graded irreducible submodule of D, $g, h \in \mathfrak{S}$ such that $I_g J_h\, C \subseteq V$, then $I_g\, GSec(C) \subseteq V$ or $J_h\, GSec(C) \subseteq V$ or $I_g J_h \subseteq Ann_A(C)$.*

Proof. Assume that $I_g\, GSec(C) \nsubseteq V$ and $J_h\, GSec(C) \nsubseteq V$. So we need to show that $I_g J_h \subseteq Ann_A(C)$. Let $b_g \in I_g$ and $e_h \in J_h$. By assumption, there exists $c_g \in I_g$ such that $c_g\, GSec(C) \nsubseteq V$ but $c_g J_h C \subseteq V$. Now by Lemma 3.3, shows that $c_g J_h \subseteq Ann_A(C)$ and so $(I_g \backslash (V :_{h(A)} GSec(C)))J_h \subseteq Ann_A(C)$. Similarly, there exists $y_h \in (J_h \backslash (V :_{h(A)} GSec(C)))$ such that $I_g y_h \subseteq Ann_A(C)$ and $I_g(J_h \backslash (V :_{h(A)} GSec(C))) \subseteq Ann_A(C)$. Thus we have $c_g y_h \in Ann_A(C)$, $c_g e_h \in Ann_A(C)$ and $b_g y_h \in Ann_A(C)$. Since $c_g + b_g \in I_g$ and $y_h + e_h \in J_h$, we have $(c_g + b_g)(y_h + e_h)C \subseteq V$. Thus either $(c_g + b_g)GSec(C) \subseteq V$ or $(y_h + e_h)GSec(C) \subseteq V$ or $(c_g + b_g)(y_h + e_h) \in Ann_A(C)$. If $(c_g + b_g)GSec(C) \subseteq V$, then $b_g GSec(C) \nsubseteq V$. Thus $b_g \in (I_g \backslash (V :_{h(A)} GSec(C)))$ which implies that $b_g e_h \in Ann_A(C)$. Similarly, if $(y_h + e_h)GSec(C) \subseteq V$, then we have $b_g e_h \in Ann_A(C)$. If $(c_g + b_g)(y_h + e_h) \in Ann_A(C)$, then $(c_g y_h + c_g e_h + b_g y_h + b_g e_h) \in Ann_A(C)$ so $b_g e_h \in Ann_A(C)$. Hence $I_g J_h \subseteq Ann_A(C)$. □

Theorem 3.5. *Let $C \leq^{sub}_{\mathfrak{S}} D$, $I = \bigoplus_{g\in\mathfrak{S}} I_g \leq^{id}_{\mathfrak{S}} A$ and $J = \bigoplus_{g\in\mathfrak{S}} J_g \leq^{id}_{\mathfrak{S}} A$. Then the following statements are equivalent:*

(a) *If $r_g t_h\, C \subseteq V_1 \cap V_2$ for some $r_g, t_h \in h(A)$ and completely graded irreducible submodules V_1, V_2 of D, then $r_g\, GSec(C) \subseteq V_1 \cap V_2$ or $t_h\, GSec(C) \subseteq V_1 \cap V_2$ or $r_g t_h \in Ann_A(C)$.*

(b) *$I_g J_h\, C \subseteq U$ for some $U \leq^{sub}_{\mathfrak{S}} D$ and $g, h \in \mathfrak{S}$, then $I_g\, GSec(C) \subseteq U$ or $J_h\, GSec(C) \subseteq U$ or $I_g J_h \subseteq Ann_A(C)$.*

(c) *For each $r_g, t_h \in h(A)$, we have $r_g\, GSec(C) \subseteq r_g t_h C$ or $t_h\, GSec(C) \subseteq r_g t_h C$ or $r_g t_h C = 0$.*

(d) *If $r_g t_h\, C \subseteq U$, for some $U \leq^{sub}_{\mathfrak{S}} D$ and $r_g, t_h \in h(A)$, then $r_g\, GSec(C) \subseteq U$ or $t_h\, GSec(C) \subseteq U$ or $r_g t_h\, C = 0$.*

Proof. (a) ⇒ (b): Suppose that $I_g J_h\, C \subseteq U$ for some $U \leq^{sub}_{\mathfrak{S}} D$, $g, h \in \mathfrak{S}$ and $I_g J_h \nsubseteq Ann_A(C)$. By Lemma 3.4, for all completely graded irreducible submodule V of D with $U \subseteq V$ either $I_g\, GSec(C) \subseteq V$ or $J_h\, GSec(C) \subseteq V$. If $I_g\, GSec(C) \subseteq V$ (resp. $J_h\, GSec(C) \subseteq V$) for all completely graded irreducible submodule V of D with $U \subseteq V$, we are done. If there exists V_1

and V_2 two completely graded irreducible submodules of D such that $U \subseteq V_1$, $U \subseteq V_2$, $I_g\, GSec(C) \nsubseteq V_1$ and $J_h\, GSec(C) \subseteq V_2$. Then $I_g\, GSec(C) \subseteq V_2$ and $J_h\, GSec(C) \subseteq V_1$. As $I_g J_h C \subseteq V_1 \cap V_2$ we have either $I_g\, GSec(C) \subseteq V_1 \cap V_2$ or $J_h\, GSec(C) \subseteq V_1 \cap V_2$, which is a contradiction.

(b) $\Rightarrow$ (d): Assume $r_g t_h C \subseteq U$, for some $U \leq^{sub}_{\mathfrak{S}} D$ and $r_g, t_h \in h(A)$. Let $I = Ar_g$ and $J = At_h$ be two graded ideals of A generated by r_g, t_h, respectively. Then $I_g J_h C \subseteq U$. By assumption we have either $I_g\, GSec(C) \subseteq U$ or $J_h\, GSec(C) \subseteq U$ or $I_g J_h \subseteq Ann_A(C)$. So either $r_g\, GSec(C) \subseteq U$ or $t_h\, GSec(C) \subseteq U$ or $r_g t_h \subseteq Ann_A(C)$.

(d) $\Rightarrow$ (c): Let $r_g, t_h \in h(A)$, then we have $r_g t_h C \subseteq r_g t_h C$. By assumption, either $r_g\, GSec(C) \subseteq r_g t_h C$ or $t_h\, GSec(C) \subseteq r_g t_h C$ or $r_g t_h C = 0$.

(c) $\Rightarrow$ (a): Let $r_g, t_h \in h(A)$ and V_1, V_2 are completely graded irreducible submodules of D such that $r_g t_h\, C \subseteq V_1 \cap V_2$. By assumption, either $r_g\, GSec(C) \subseteq r_g t_h C \subseteq V_1 \cap V_2$ or $t_h\, GSec(C) \subseteq r_g t_h C \subseteq V_1 \cap V_2$ or $r_g t_h C = 0$, as needed. □

Definition 3.6. A graded submodule C of D is called graded strongly 2-absorbing secondary (briefly, gr-2^{abs}_{st}-secondary) submodule of D if satisfies the equivalent conditions of Theorem 3.5. By gr-2^{abs}_{st}-secondary module, we mean a graded module which is gr-2^{abs}_{st}-secondary submodule of itself.

Clearly every graded gr-2^{abs}_{st}-secondary submodule is gr-2^{abs}-secondary submodule. In general, the converse is not true as shown in the following example.

Example 3.7. Let $A = \mathbb{Z}$ and $\mathfrak{S} = \mathbb{Z}_2$. Then A is a $\mathfrak{S}$-graded ring with $A_0 = \mathbb{Z}$ and $A_1 = \{0\}$. Consider $D = \mathbb{Z}_6 \oplus \mathbb{Q}$ as a $\mathbb{Z}$-module, D is a $\mathfrak{S}$-graded module with $D_0 = \mathbb{Z}_6 \oplus 0$ and $D_1 = \bar{0} \oplus \mathbb{Q}$. Then D is gr-2^{abs}-secondary module. But D is not gr-2^{abs}_{st}-secondary module since $0 \neq 6D \subseteq \bar{0} \oplus \mathbb{Q}$, $GSec(D) = D$, $2D \nsubseteq \bar{0} \oplus \mathbb{Q}$ and $3D \nsubseteq \bar{0} \oplus \mathbb{Q}m$.

Theorem 3.8. *Let C be gr-2^{abs}_{st}-second submodule of D. Then C is gr-2^{abs}_{st}-secondary submodule of D.*

Proof. Let $r_g, t_h \in h(A)$ and $U \leq^{sub}_{\mathfrak{S}} D$ with $r_g t_h\, C \subseteq U$. Then $r_g C \subseteq U$ or $t_h C \subseteq U$ or $r_g t_h C = 0$ by (Theorem 3.2, [1]). Since $GSec(C) \subseteq C$ by (Lemma 4.6, [14]), then either $r_g GSec(C) \subseteq r_g C \subseteq U$ or $t_h GSec(C) \subseteq t_h C \subseteq U$ or $r_g t_h C = 0$. □

Theorem 3.9. *Let D be a gr-com A-module and $C \leq^{sub}_{\mathfrak{S}} D$. If $Ann_A(C)$ is gr-2^{abs}-primary ideal of A, then C is gr-2^{abs}_{st}-secondary submodule of D.*

Proof. Let $r_g, t_h \in h(A)$ and $U \leq^{sub}_{\mathfrak{S}} D$ such that $r_g t_h C \subseteq U$. Since D is gr-com module, there exists $I \leq^{id}_{\mathfrak{S}} A$ with $U = (0 :_D I)$. Thus $r_g t_h C \subseteq (0 :_D I)$, then $r_g t_h I \subseteq Ann_A(C)$. If $r_g t_h \in Ann_A(C)$, we are done. If $r_g t_h \notin Ann_A(C)$, then either $Ir_g \subseteq Grad(Ann_A(C))$ or $It_h \subseteq Grad(Ann_A(C))$ by (Proposition 2.28, [16]). If $Ir_g \subseteq Grad(Ann_A(C))$. By (Lemma 4.6, [14]), we have $Ir_g \subseteq Grad(Ann_A(C)) \subseteq Ann_A(GSec(C))$, thus $Ir_g GSec(C) = 0$ and so $r_g GSec(C) \subseteq (0 :_D I) = U$ since D is gr-com module. Similarly, if $It_h \subseteq Grad(Ann_A(C))$, we get $t_h GSec(C) \subseteq U$. □

Example 3.10. Let $A = \mathbb{Z}$ and $\mathfrak{S} = \mathbb{Z}_3$. Then A is a $\mathfrak{S}$-graded ring with $A_0 = \mathbb{Z}$, $A_1 = \{0\}$ and $A_2 = \{0\}$. Consider $D = \mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Q}$ as a $\mathbb{Z}$-module, D is a $\mathfrak{S}$-graded module with $D_0 = \mathbb{Z}_p \oplus \bar{0} \oplus 0$, $D_1 = \bar{0} \oplus \mathbb{Z}_q \oplus 0$ and $D_3 = \bar{0} \oplus \bar{0} \oplus \mathbb{Q}$. D is not gr-com $\mathbb{Z}$-module since $N = \mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Z}$ is graded submodule of D and $Ann_A(N) = 0$, but $N \neq (0 :_D Ann_A(N)) = \mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Q} = D$. $Ann_A(D) = 0$ is gr-2^{abs}-primary ideal of A. But D is not gr-2^{abs}_{st}-secondary module since $0 \neq pqD \subseteq \bar{0} \oplus \bar{0} \oplus \mathbb{Q}$, $GSec(D) = D$, $pD \nsubseteq \bar{0} \oplus \bar{0} \oplus \mathbb{Q}$ and $qD \nsubseteq \bar{0} \oplus \bar{0} \oplus \mathbb{Q}$.

Lemma 3.11. *Let $C \leq^{sub}_{\mathfrak{S}} D$ and $U \leq^{sub}_{\mathfrak{S}} D$. Then:*

(a) $GSec(C) = GSec(GSec(C))$,

(b) $GSec((0 :_D I^n)) = GSec((0 :_D I))$ *for every positive integer n and* $I \leq^{id}_{\mathfrak{S}} A$.

(c) *If S is gr-P-second submodule of D such that $S \subseteq C+U$ and $Ann_A(C) \nsubseteq P$, then $S \subseteq U$.*

Proof. (a): By (Lemma 4.6 (1) and (2), [14]), we have $GSec(GSec(C)) \subseteq GSec(C)$. Conversely, let H be gr-second submodule of D such that $H \subseteq C$. Since $GSec(C) = \sum_{P \in GSpec^s(D) | P \subseteq C} P$, then $H \subseteq GSec(C)$, so $GSec(C) \subseteq GSec(GSec(C))$.

(b): Let V be gr-second submodule of D such that $V \subseteq (0 :_D I^n)$, then $I^n \subseteq Ann_A(V)$ and since $Ann_A(V)$ is gr-prime ideal of A (by Lemma 2.5, [14]), we have $I \subseteq Ann_A(V)$. Thus $V \subseteq (0 :_D I)$ and so $GSec((0 :_D I^n)) \subseteq GSec((0 :_D I))$. Conversely, let U be gr-second submodule such

that $U \subseteq (0 :_D I)$, so $UI = 0$ and so $UI^n = 0$. Hence $U \subseteq (0 :_D I^n)$, and so $GSec((0 :_D I)) \subseteq GSec((0 :_D I^n))$.

(c): Since $Ann_A(C) \nsubseteq P = Ann_A(S)$, there exists $r_\lambda \in h(A)$ such that $r_\lambda C = 0$ and $r_\lambda S \neq 0$. Since S is gr-P-second submodule of D, by (Theorem 2.6,[11]), we get $r_\lambda S = S$. Since $r_\lambda S \subseteq r_\lambda C + r_\lambda U$ and $U \leq^{sub}_{\mathfrak{S}} D$, then we have $S \subseteq r_\lambda U \subseteq U$. Therefore, $S \subseteq U$. □

Theorem 3.12. *Let $C \leq^{sub}_{\mathfrak{S}} D$. Then*

(i) *If $GSec(C)$ is a gr-2^{abs}-secondary (resp. gr-2^{abs}_{st}- secondary) submodule of D, then $GSec(C)$ is a gr-2^{abs}- second (resp. gr-2^{abs}_{st}-second) submodule of D.*

(ii) *If C is a gr-second radical submodule of D, then C is a gr-2^{abs}- second (resp. gr-2^{abs}_{st}-second) submodule of D if and only if C is a gr-2^{abs}-secondary (resp. gr-2^{abs}_{st}- secondary) submodule of D.*

Proof. (i): Let $r_g, t_h \in h(A)$ and U be a graded completely irreducible submodule of D such that $r_g t_h\, GSec(C) \subseteq U$. As $GSec(C)$ is a gr-2^{abs}-secondary submodule of D, we have either $r_g\, GSec(GSec(C)) \subseteq U$ or $t_h\, GSec(GSec(C)) \subseteq U$ or $r_g t_h \in Ann_A(GSec(C))$. By Lemma 3.11 (a), we get either $r_g\, GSec(C) \subseteq U$ or $t_h\, GSec(C) \subseteq U$ or $r_g t_h \in Ann_A(GSec(C))$, hence $GSec(C)$ is a gr-2^{abs}-second submodule of D, as required.

(ii): This follows from part (i). □

Definition 3.13. Let C and U be two graded submodules of D. We define the graded coproduct of C and U by $(0 :_D Ann_A(C)Ann_A(U))$ and denoted by $C^g(CU)$.

Theorem 3.14. *Let $C \leq^{sub}_{\mathfrak{S}} D$ such that $GSec(C)$ be gr-second submodule of D. Then:*

(a) *C is a gr-2^{abs}_{st}-secondary submodule of D.*

(b) *If D is gr-com A-module, then $C^g(C^n)$ is a gr-2^{abs}_{st}-secondary submodule of D for every positive integer $n \geq 1$, where $C^g(C^n)$ means the graded coproduct of C, n times.*

Proof. (a): Let $r_g, t_h \in h(A)$ and $U \leq^{sub}_{\mathfrak{S}} D$ such that $r_g t_h\, C \subseteq U$, and let $t_h GSec(C) \nsubseteq U$. Then since $GSec(C)$ is gr-second submodule and $r_g GSec(C) \subseteq r_g C \subseteq (U :_D t_h)$, we have $r_g GSec(C) = 0$. Hence $r_g GSec(C) \subseteq U$.

(b): Let D be gr-com A-module, there exists $I \leq^{id}_{\mathfrak{G}} A$ such that $C = (0 :_D I)$. Hence by Lemma 3.11 (b),

$$GSec(C^g(C^n)) = GSec((0 :_D I^n)) = GSec((0 :_D I)) = GSec(C).$$

Now the results follows from part (a). □

Lemma 3.15. *Let D graded A-module. Then the following hold:*

(a) *Let $I \leq^{id}_{\mathfrak{G}} A$ and $C \leq^{sub}_{\mathfrak{G}} D$. If S is gr-P-second submodule of D such that $S \subseteq (0 :_D I) + C$, then $S \subseteq (0 :_D I)$ or $S \subseteq C$.*

(b) *Let C and U be two graded submodules of D such that whenever $S \subseteq C + U$, we have $S \subseteq C$ or $S \subseteq U$ for every gr-second submodule S of D. Then $GSec(C + U) = GSec(C) + GSec(U)$.*

(c) *Let D be gr-com A-module, if S is gr-P-second submodule of D such that $S \subseteq C + U$ for every pair of graded submodules C and U of D, then either $S \subseteq C$ or $S \subseteq U$, consequently $GSec(C + U) = GSec(C) + GSec(U)$.*

Proof. (a): If $Ann_A((0 :_D I)) \subseteq P$, then $I \subseteq P$. Thus $(0 :_D P) \subseteq (0 :_D I)$ and so $S \subseteq (0 :_D Ann_A(S)) = (0 :_D P) \subseteq (0 :_D I)$. If $Ann_A((0 :_D I)) \nsubseteq P$, then $S \subseteq C$ by Lemma 3.11 (c).

(b): If $GSec(C + U) = 0$, then $GSec(C) + GSec(U) = 0$, and so $GSec(C+U) = GSec(C)+GSec(U)$. If $GSec(C+U) \neq 0$, then there exists a graded second submodule S of D. By our assumption, $S \subseteq C$ or $S \subseteq U$, thus $S \subseteq GSec(C)$ or $S \subseteq GSec(U)$. Hence $GSec(C + U) \subseteq GSec(C) + GSec(U)$. Now, let $S_1 \subseteq C$ and $S_2 \subseteq U$ be two gr-second submodule of D. Then $S_1 \subseteq C+U$ and S_1 is gr-second submodule of D and $S_2 \subseteq C+U$ and S_2 is gr-second submodule of D. Thus $GSec(C)+GSec(U) \subseteq GSec(C+U)$.

(c): This follows from parts (a) and (b). □

Lemma 3.16. *Let $I \leq^{id}_{\mathfrak{G}} A$ and let D be a graded finitely generated A-module. Let $x \in h(A)$ such that $xD \subseteq ID$, then $(x^n + y)D = 0$ for some $y \in I \cap h(A)$.*

Proof. Let $a_1, a_2, ..., a_n$ be a homogenous elements generate D. Since $xD \subseteq ID$, we have $xa_i = \sum_{j=1}^{n} y_{ij}a_j$ for some element $y_{ij} \in I$. The equation can be written in the form $\sum_{j=1}^{n}(\rho_{ij} - y_{ij})a_j = 0$, where $\rho_{ij} = I_n x$, I_n is n identity matrix. So we have $(xI_n - Y) = 0$, where $Y = (y_{ij})$ and a is

column vector whose coefficients are a_j. Multiply by the adjoint of $xI_n - Y$, gives that $det(xI_n - Y)a = 0$, thus $det(xI_n - Y)a_i = 0$ for all i. Hence, $det(xI_n - Y)D = 0$ and $det(xI_n - Y)$ is of the form $(x^n + y)$, for some element $y \in I$. □

Theorem 3.17. *Let D graded A-module. Then:*

(a) *If D is a graded finitely generated comultiplication module and P is gr-prime ideal of A containing $Ann_A(D)$, then $(0 :_D P)$ is gr-second submodule of D.*

(b) *If D is a graded finitely generated comultiplication module, then the natural map ϕ of $GSpec^s(D)$ is surjective.*

(c) *If the natural map ϕ of $GSpec^s(D)$ is surjective and $I \leq^{id}_{\mathfrak{S}} A$ containing $Ann_A(D)$, then $Ann_A((0 :_D Grad(I))) = Grad(I)$.*

Proof. (a): Using (Proposition 3.15 (b), [10]), it is sufficient to show that $Ann_A((0 :_D P)) = P$. Let $r_g \in h(A)$ such that $r_g \in Ann_A((0 :_D P))$, then $(0 :_D P) \subseteq (0 :_D r_g)$. Since D is a gr-com module, it is follows that

$$D \subseteq ((0 :_D P) :_D Ann_A(PD)) \subseteq ((0 :_D r_g) :_D Ann_A(PD)) = (PD :_D r_g).$$

Thus $r_g D \subseteq PD$, by Lemma 3.16, we have $(r_g^n + a)D = 0$, for some $n \in \mathbb{Z}^+$ and $a \in P$. As $Ann_A(D) \subseteq P$, $r_g^n \in P$ and so $r_g \in Grad(P) = P$ since P is gr-prime ideal of A. Therefore, $Ann_A((0 :_D P)) \subseteq P$. The reverse inclusion is clear.

(b): It is follow from part (a).

(c): Note that $Ann_A(D) \subseteq I \subseteq Grad(I) = \cap_i P_i$, where P_i runs through $W(I) = \{P \in Spec(A) | I \subseteq P\}$. Since ϕ is surjective and $Ann_A(D) \subseteq P_i$ for each $P_i \in W(I)$, we have $P_i = Ann_A(Q_i)$ for some $Q_i \in GSpec^s(D)$. Hence,

$$\begin{aligned} Grad(I) \subseteq Ann_A((0 :_D Grad(I))) &= Ann_A((0 :_D \cap_i P_i)) \\ &\subseteq \cap_i Ann_A((0 :_D Ann_A(Q_i))) = \cap_i P_i = Grad(I). \end{aligned}$$

Thus $Ann_A(0 :_D Grad(I)) = Grad(I)$. □

Lemma 3.18. *Let D be a graded finitely generated comultiplication A-module D, $C \leq^{sub}_{\mathfrak{S}} D$ and $U \leq^{sub}_{\mathfrak{S}} D$. Then*

(a) $GSec(C) = (0 :_D Grad(Ann_A(C)))$.

(b) $Ann_A(GSec(C)) = Grad(Ann_A(C))$.

(c) *If C and U are gr-secondary submodules of D with $GSec(C) = GSec(U)$, then $C + U$ is graded secondary submodule of D.*

Proof. (a): By (Lemma 4.6, [14]), we have $GSec(C) \subseteq (0 :_D Grad(Ann_A(C)))$. Since D is gr-com module, $GSec(C) = (0 :_D Ann_A(GSec(C)))$, so it is enough to show that $Ann_A(GSec(C)) \subseteq Grad(Ann_A(C))$. Let P be gr-prime ideal such that $Ann_A(C) \subseteq P$. As P is gr-prime ideal containing $Ann_A(D)$, then $(0 :_D P)$ is gr-second submodule of D contained in C, using Theorem 3.17 (a). Thus

$$(0 :_D P) \subseteq GSec(C) = (0 :_D Ann_A(GSec(C)).$$

Hence by similar arguments as in Theorem 3.17 (a), we have

$$Ann_A(GSec(C) \subseteq P,$$

consequently, $Ann_A(GSec(C)) \subseteq Grad(Ann_A(C))$.

(b): By Theorem 3.17 (c), we have $Ann_A(0 :_D Grad(Ann_A(C))) = Grad(Ann_A(C))$, the result follows from part (a).

(c): Let C and U are gr-secondary submodules, then the graded ideals $Grad(Ann_A(C))$ and $Grad(Ann_A(U))$ are gr-primes. In addition,

$$Grad(Ann_A(C)) = Ann_A(GSec(C)) = Ann_A(GSec(U)) = Grad(Ann_A(U))$$

. If we call $Grad(Ann_A(C)) = P$ gr-prime ideal, then C and U are gr-P-secondary submodules and so $C + U$ is gr-P-secondary submodule. □

Corollary 3.19. *Let D be a graded finitely generated comultiplication A-module, If S is gr-secondary submodule of D, then $GSec(S)$ is gr-second submodule of D.*

Proof. By Lemma 3.18 (a), $GSec(S) = (0 :_D Grad(Ann_A(S)))$. Now, since $Grad(Ann_A(S))$ is gr-prime ideal containing $Ann_A(D)$, then the result follows from Theorem 3.17. Therefore, $GSec(S)$ is gr-second submodules of D. □

Theorem 3.20. *Let D be a graded finitely generated comultiplication A-module and $C \leq^{sub}_{\mathfrak{S}} D$. If C is gr-2^{abs}_{st}-secondary submodule of D, then $Ann_A(C)$ is gr-2^{abs}-primary ideal of A.*

Proof. Let $x_g, y_h, z_\lambda \in h(A)$, $x_g y_h z_\lambda \in Ann_A(C)$, $x_g z_\lambda \notin Grad(Ann_A(C))$ and $y_h z_\lambda \notin Grad(Ann_A(C))$. We show that $x_g y_h \in Ann_A(C)$. Using Lemma 3.18, $Ann_A(GSec(C)) = Grad(Ann_A(C))$, so there exists completely graded irreducible submodules V_1,V_2 of D such that $x_g z_\lambda(GSec(C)) \nsubseteq V_1$ and $y_h z_\lambda(GSec(C)) \nsubseteq V_2$. But $x_g y_h z_\lambda C = 0 \subseteq V_1 \cap V_2$, then $x_g y_h C \subseteq (V_1 \cap V_2 :_D z_\lambda)$. Since C is gr-2^{abs}_{st}-secondary submodule of D, then either $x_g GSec(C) \subseteq (V_1 \cap V_2 :_D z_\lambda)$ or $y_h GSec(C) \subseteq (V_1 \cap V_2 :_D z_\lambda)$ or $x_g y_h C = 0$. If $x_g GSec(C) \subseteq (V_1 \cap V_2 :_D z_\lambda)$, then we have $x_g z_\lambda(GSec(C)) \subseteq V_1$, a contradiction. If $y_h GSec(C) \subseteq (V_1 \cap V_2 :_D z_\lambda)$, then $y_h z_\lambda(GSec(C)) \subseteq V_2$, a contradiction. Thus $x_g y_h C = 0$. □

Theorem 3.21. *Let D be a gr-com A-module. Then:*

(a) *If $C_1, C_2, ...C_m$ are gr-2^{abs}_{st}-secondary submodules of D with the same graded second radical, then $C = \sum_{j=1}^{m} C_j$ is gr-2^{abs}_{st}-secondary submodule of D.*

(b) *If $C_1, C_2, ...C_m$ are gr-2^{abs}-secondary submodules of D with the same graded second radical, then $C = \sum_{j=1}^{m} C_j$ is gr-2^{abs}-secondary submodule of D.*

(c) *If C_1 and C_2 are two gr-secondary submodules of D, then $C_1 + C_2$ is gr-2^{abs}_{st}-secondary submodule of D.*

(d) *If D is graded finitely generated, $C \leq^{sub}_{\mathfrak{S}} D$ which possess a gr-secondary representation and $GSec(C) = U_1 + U_2$, where U_1 and U_2 are two gr-minimal submodules of D, then C is gr-2^{abs}_{st}-secondary submodule of D.*

Proof. (a): Let $r_g, t_h \in h(A)$ and $U \leq^{sub}_{\mathfrak{S}} D$ such that $r_g t_h C \subseteq U$. Then for each $j = 1, 2, ...m$, $r_g t_h C_j \subseteq U$. If there exists $1 \leq i \leq m$ such that $r_g GSec(C_i) \subseteq U$ or $t_h GSec(C_i) \subseteq U$, then $r_g GSec(C) \subseteq U$ or $t_h GSec(C) \subseteq U$ since $GSec(C) = GSec(\sum_{j=1}^{m} C_j) = \sum_{j=1}^{m} GSec(C_j) = GSec(C_j)$, for each $j = 1, 2, ...m$, using Lemma 3.15. Otherwise, $r_g t_h C_j = 0$ for each $j = 1, 2, ...m$, thus $r_g t_h C = 0$.

(b): Similar to part (a).

(c): Since C_1 and C_2 are two gr-secondary submodules of D, then $Ann_A(C_1)$ and $Ann_A(C_2)$ are gr-primary ideals of A. Thus $Ann_A(C_1 + C_2) = Ann_A(C_1) \cap Ann_A(C_2)$ is gr-2^{abs}-primary ideal of A by (Lemma 2.9, [7]). Therefore by Theorem 3.9, $C_1 + C_2$ is gr-2^{abs}_{st}-secondary submodule of D.

(d): Let $C = \sum_{j=1}^{m} C_j$ be a gr-secondary representation. Then by Lemma 3.15, $GSec(C) = \sum_{j=1}^{m} GSec(C_j)$, since $GSec(C_j)$'s are gr-second submodules of D by Corollary 3.19, we have

$$\{GSec(C_1), GSec(C_2), ..., GSec(C_m)\} = \{U_1, U_2\}.$$

Now since U_1 and U_2 are gr-minimal submodules of D, without loss of generality, we may assume that for some $1 \leq i \leq m$, $\{GSec(C_1), ..., GSec(C_i)\} = \{U_1\}$ and $\{GSec(C_{i+1}), ..., GSec(C_m)\} = \{U_2\}$. Put $W_1 := C_1 + ... + C_i$ and $W_2 := C_{i+1} + ... + C_m$. Using Lemma 3.18, W_1 and W_2 are gr-secondary submodules of D. Thus by part (c), $C = W_1 + W_2$ is gr-2_{st}^{abs}-secondary submodule of D. □

As shown in the example below, the direct sum of two gr-2_{st}^{abs}-secondary A-modules is not a gr-2_{st}^{abs}-secondary A-module in general

Example 3.22. Let $A = \mathbb{Z}$ and $\mathfrak{S} = \mathbb{Z}_2$, then A is a $\mathfrak{S}$-graded ring with $A_0 = \mathbb{Z}$ and $A_1 = \{0\}$. The $\mathbb{Z}$-modules $\mathbb{Z}_6$ and $\mathbb{Z}_{10}$ are gr-2_{st}^{abs}-secondary $\mathbb{Z}$-modules. Let $D = \mathbb{Z}_6 \oplus \mathbb{Z}_{10}$ is $\mathfrak{S}$-graded module with $D_0 = \mathbb{Z}_6 \oplus \bar{0}$ and $D_1 = \bar{0} \oplus \mathbb{Z}_{10}$. Then D is not gr-2_{st}^{abs}-secondary $\mathbb{Z}$-modules since $0 \neq (2)(3)D \subseteq \bar{0} \oplus \mathbb{Z}_{10}$, but $(2)D \nsubseteq \bar{0} \oplus \mathbb{Z}_{10}$ and $(3)D \nsubseteq \bar{0} \oplus \mathbb{Z}_{10}$.

Lemma 3.23. *Let $\varphi : D \to D'$ be a graded monomorphism of graded A-modules. Then the following hold:*

(i) *If $C \leq_{\mathfrak{S}}^{sub} D$, then $GSec(\varphi(C)) = \varphi(GSec(C))$.*

(ii) *If $C' \leq_{\mathfrak{S}}^{sub} D'$ such that $C' \subseteq \varphi(D)$, then $GSec(\varphi^{-1}(C'^{-1}(GSec(C'))$.*

Proof. (i): Let Q' be gr-second submodule of $\varphi(C)$. Then $\varphi^{-1}(Q')$ is gr-second submodule of C. So $\varphi(\varphi^{-1}(Q')) \subseteq \varphi(GSec(C))$. Thus $GSec(\varphi(C)) \subseteq \varphi(GSec(C))$. Now let Q be gr-second submodule of C. Then $\varphi(Q)$ is gr-second submodule of $\varphi(C)$. So $\varphi(Q) \subseteq GSec(\varphi(C))$. Thus $\varphi(GSec(C)) \subseteq GSec(\varphi(C))$.

(ii): Let Q be gr-second submodule of $\varphi^{-1}(C')$. Then $\varphi(Q)$ is gr-second submodule of C'. Hence $\varphi^{-1}(\varphi(Q)) \subseteq \varphi^{-1}(GSec(C'))$ and so $GSec(\varphi^{-1}(C'^{-1}(GSec(C'))$. For the reverse inclusion, let Q' be gr-second submodule of C'. Then $\varphi^{-1}(Q')$ is gr-second submodule of $\varphi^{-1}(C')$. Hence $\varphi^{-1}(GSec(C'^{-1}(C'))$. □

Theorem 3.24. *Let $\varphi : D \to D'$ be a graded monomorphism of graded A-modules. Then the following hold:*

(a) *If C is gr-2^{abs}_{st}-secondary submodule of D, then $\varphi(C)$ is gr-2^{abs}_{st}-secondary submodule of D'.*

(b) *If C' is gr-2^{abs}_{st}-secondary submodule of D' and $C' \subseteq \varphi(D)$, then $\varphi^{-1}(C')$ is gr-2^{abs}_{st}-secondary submodule of D.*

Proof. (a): As $C \neq 0$ and φ is graded monomorphism, $\varphi(C) \neq 0$. Let $r_g, t_h \in h(A)$ and U' be a graded submodule of D' with $r_g t_h \varphi(C) \subseteq U'$. Thus $r_g t_h C \subseteq \varphi^{-1}(U')$. Since C is gr-2^{abs}_{st}-secondary submodule of D, either $r_g GSec(C) \subseteq \varphi^{-1}(U')$ or $t_h GSec(C) \subseteq \varphi^{-1}(U')$ or $r_g t_h C = 0$. By Lemma 3.23 (i),

$$r_g GSec(\varphi(C)) = r_g \varphi(GSec(C)) \subseteq \varphi(\varphi^{-1}(U')) = \varphi(D) \cap U' \subseteq U'$$

or

$$t_h GSec(\varphi(C)) = t_h \varphi(GSec(C)) \subseteq \varphi(\varphi^{-1}(U')) = \varphi(D) \cap U' \subseteq U'$$

or $r_g t_h \varphi(C) = 0$.

(b): If $\varphi^{-1}(C') = 0$, then $\varphi(D) \cap C'^{-1}(C')) = \varphi(0) = 0$. So $C' = 0$, a contradiction. Hence $\varphi^{-1}(C') \neq 0$. Let $r_g, t_h \in h(A)$ and $U \leq^{sub}_{\mathfrak{S}} D$ with $r_g t_h \varphi^{-1}(C') \subseteq U$, then $r_g t_h C' = r_g t_h (\varphi(D) \cap C') = r_g t_h \varphi(\varphi^{-1}(C')) \subseteq \varphi(U)$. Since C' is gr-2^{abs}_{st}-secondary submodule of D', either $r_g GSec(C') \subseteq \varphi(U)$ or $t_h GSec(C') \subseteq \varphi(U)$ or $r_g t_h C' = 0$. By Lemma 3.23(ii),

$$r_g GSec(\varphi^{-1}(C')) = r_g \varphi^{-1}(GSec(C'^{-1}(\varphi(U)) = U$$

or

$$t_h GSec(\varphi^{-1}(C')) = t_h \varphi^{-1}(GSec(C'^{-1}(\varphi(U)) = U$$

or $r_g t_h^{-1} \varphi(C') = 0$. □

Corollary 3.25. *Let $C \subseteq U$ be two graded submodules of D. Then C is gr-2^{abs}_{st}-secondary submodule of U if and only if C is gr-2^{abs}_{st}-secondary submodule of D.*

Proof. Using the natural monomorphism $U \to D$, this follows from Theorem 3.24. □

Lemma 3.26. *Let D be a gr-com A-module with unique gr-minimal submodule U and $C \leq^{sub}_{\mathfrak{S}} D$.*

(a) *If S is gr-second submodule of C, then S/U is gr-second submodule of C/U.*

(b) $GSec(C)/U \subseteq GSec(C/U)$.

Proof. (a): Assume that S is a gr-second submodule of C and $r_g \in h(A)$ with $r_g(S/U) \subseteq (H/U)$, where $H/U \leq^{sub}_{\mathfrak{S}} C/U$. Then $r_gS \subseteq H$, and $H \leq^{sub}_{\mathfrak{S}} C$. As S is gr-second submodule of C, then we have $r_gS = 0$ or $S \subseteq H$. If $r_gS = 0$, then $r_g(S/U) = U$. If $S \subseteq H$, then $(S/U) \subseteq (H/U)$. Therefore, S/U is gr-second submodule of C/U.

(b): This follows from part (a). □

Theorem 3.27. *Let D be a gr-com A-module with unique gr-minimal submodule U and $C \leq^{sub}_{\mathfrak{S}} D$ such that $x_\alpha C \neq U$ for each $x_\alpha \in h(A)$. If C/U is gr-2^{abs}_{st}-secondary submodule of D/U, then C is gr-2^{abs}_{st}-secondary submodule of D.*

Proof. Let $r_g, t_h \in h(A)$ and $H \leq^{sub}_{\mathfrak{S}} D$ such that $r_gt_h\,C \subseteq H$. Then $r_gt_h(C/U) \subseteq H/U$ implies either $r_g\,GSec(C/U) \subseteq H/U$ or $t_h\,GSec(C/U) \subseteq H/U$ or $r_gt_h(C/U) = 0$. If $r_gt_h(C/U) = 0$, then $r_gt_h\,C = 0$ since $x_\alpha C \neq U$ for each $x_\alpha \in h(A)$. Now by Lemma 3.26, we have $GSec(C)/U \subseteq GSec(C/U)$, so if $r_g\,GSec(C/U) \subseteq H/U$ or $t_h\,GSec(C/U) \subseteq H/U$, then either $r_g\,GSec(C) \subseteq H$ or $t_h\,GSec(C) \subseteq H$, as needed. □

Let A_1 and A_2 be $\mathfrak{S}$-graded rings, $A = A_1 \times A_2$ is $\mathfrak{S}$-graded ring with $A_g = (A_1)_g \times (A_2)_g$ for all $g \in \mathfrak{S}$. Let D_1 be a $\mathfrak{S}$-graded A_1-module, D_2 be a $\mathfrak{S}$-graded A_2-module and $A = A_1 \times A_2$. Then $D = D_1 \times D_2$ is a $\mathfrak{S}$-graded A-module with $D_g = (D_1)_g \times (D_2)_g$ for all $g \in \mathfrak{S}$. Furthermore, each graded submodule of D is of the form $C = C_1 \times C_2$, where $C_i \leq^{sub}_{\mathfrak{S}} D_i$, where $i = 1, 2$, see [13].

Theorem 3.28. *Let D_1 be a graded A_1-module, D_2 be a graded A_2-module and $A = A_1 \times A_2$ be a graded ring. Let $D = D_1 \times D_2$ be a graded A-module. Then D_1 and D_2 are gr-com modules if and only if $D = D_1 \times D_2$ is gr-com module.*

Proof. Let D_1 and D_2 be gr-com modules and let $N = N_1 \times N_2 \leq^{sub}_{\mathfrak{S}} D$, where $N_i \leq^{sub}_{\mathfrak{S}} D_i$ $(i = 1, 2)$. Then $N_1 = (0 :_{D_1} I_1)$ and $N_2 = (0 :_{D_2} I_2)$, where I_1 and I_2 are graded ideals of A_1 and A_2, respectively. Now $N_1 \times N_2 = (0 :_D I_1 \times I_2)$, that is, D is gr-com module. Conversely, if D is gr-com module and $N_1 \leq^{sub}_{\mathfrak{S}} D_1$, then $N_1 \times 0 = (0 :_D I \times J)$ for graded ideals I and J of A_1 and A_2, respectively. Therefore, $N_1 = (0 :_{D_1} I)$, as needed. Similarly D_2 is gr-com module. □

Theorem 3.29. *Let $A = A_1 \times A_2$ is $\mathfrak{S}$-graded ring and $D = D_1 \times D_2$ be a graded A-module, where D_i is a graded A_i-module, $i = 1, 2$. If $C = C_1 \times C_2$ is gr-second submodule of D. Then*

(a) *C is a gr-second submodule of D if and only if $C = C_1 \times 0$ or $C = 0 \times C_2$ where C_1 is a gr-second submodule of D_1 and C_2 is a gr-second submodule of D_2.*

(b) *$GSec(C) = GSec(C_1) \times GSec(C_2)$.*

Proof. (a): Assume that $C = C_1 \times C_2$ is gr-second submodule of D, then $Ann_A(C)$ is gr-prime ideal of A. As $Ann_A(C) = Ann_{A_1}(C_1) \times Ann_{A_2}(C_2)$ is gr-prime ideal, then either $Ann_{A_2}(C_2) = A_2$ and $Ann_{A_1}(C_1)$ is gr-prime ideal of A_1 or $Ann_{A_1}(C_1) = A_1$ and $Ann_{A_2}(C_2)$ is gr-prime ideal of A_2. If $Ann_{A_2}(C_2) = A_2$ and $Ann_{A_1}(C_1)$ is gr-prime ideal of A_1, then $C_2 = 0$ and $C = C_1 \times 0$ is gr-second submodule of D implies that C_1 is gr-second submodule of D_1. Similarly if $Ann_{A_1}(C_1) = A_1$ and $Ann_{A_2}(C_2)$ is gr-prime ideal of A_2, then $C_1 = 0$ and C_2 is gr-second submodule of D_2. The converse is clear.

(b): This follows from part (a). □

Theorem 3.30. *Let $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2$ and $A = A_1 \times A_2$ is $\mathfrak{S}$-graded ring where A_i is $\mathfrak{S}_i$-graded ring with $rank(\mathfrak{S}_i) = 2$, for $i = 1, 2$. Let $D = D_1 \times D_2$ be a graded A-module where D_1 is gr-com A_1-module and D_2 is gr-com A_2-module. then the following hold:*

(a) *If D_1 is finitely generated A_1-module, then a non-zero graded submodule U_1 of D_1 is gr-2^{abs}_{st}-secondary submodule if and only if $C = U_1 \times 0$ is gr-2^{abs}_{st}-secondary submodule of D.*

(b) *If D_2 is finitely generated A_2-module, then a non-zero graded submodule U_2 of D_2 is gr-2^{abs}_{st}-secondary submodule if and only if $C = 0 \times U_2$ is gr-2^{abs}_{st}-secondarysubmodule of D.*

(c) *If U_1 is gr-secondary submodule of D_1 and U_2 is gr-secondary submodule of D_2, then $C = U_1 \times U_2$ is gr-2^{abs}_{st}-secondary submodule of D.*

Proof. (a): Let U_1 be gr-2^{abs}_{st}-secondary submodule of D_1. Then $Ann_{A_1}(U_1)$ is gr-2^{abs}-primary ideal of A_1, by Theorem 3.20. Now since $Ann_A(C) = Ann_{A_1}(U_1) \times A_2$, we have $Ann_A(C)$ is a gr-2^{abs}-primary ideal of A by (Theorem 2.26, [16]). Hence the result follows from Theorem 3.9. For

converse, let $C = U_1 \times 0$ be gr-2^{abs}_{st}-secondary submodule of D. Then $Ann_A(C) = Ann_{A_1}(U_1) \times A_2$ is gr-2^{abs}-primary ideal of A by Theorem 3.20. Hence $Ann_{A_1}(U_1)$ is gr-2^{abs}-primary ideal of A_1, by (Theorem 2.26, [16]). Hence by Theorem 3.9, U_1 is gr-2^{abs}_{st}-secondary submodule of D_1.

(b): Similar to part (a).

(c): Let U_1 be gr-secondary submodule of D_1 and U_2 be gr-secondary submodule of D_2. Then $Ann_{A_1}(U_1)$ and $Ann_{A_2}(U_2)$ are gr-primary ideals of A_1 and A_2, respectively. Since $Ann_A(C) = Ann_{A_1}(U_1) \times Ann_{A_2}(U_2)$, $Ann_A(C)$ is gr-2^{abs}-primary ideal of A by (Theorem 2.26, [16]). By Theorem 3.9, we get the result. □

Lemma 3.31. *Let C be a graded submodule of gr-com A-module D. Then C is gr-secondary submodule if and only if $Ann_A(C)$ is gr-primary ideal of A.*

Proof. Let C be gr-secondary submodule of D and let $a_g b_h \in Ann_A(C)$, for some $a_g, b_h \in h(A)$. Then $a_g b_h\, C = 0$ and so $b_h\, C \subseteq (0 :_D a_g)$. As C is gr-secondary submodule, then $C \subseteq (0 :_D a_g)$ or $b_h^m\, C = 0$, for some positive integer m, by [9]. Therefore, $a_g \in Ann_A(C)$ or $b_h \in Grad(Ann_A(C))$. Conversely, let $t_\lambda \in h(A)$. Since D is gr-com module and $t_\lambda\, C$ is graded submodule of D, then $t_\lambda\, C = (0 :_D I)$ for some $I \leq^{id}_{\mathfrak{S}} A$. Now $t_\lambda I \subseteq Ann_A(C)$ then either $I \subseteq Ann_A(C)$ or $t^n_\lambda \in Ann_A(C)$, for some positive integer n. Then $Ann_A(t_\lambda\, C) \subseteq Ann_A(C)$ or $t^n_\lambda\, C = 0$. As D is gr-com module we have $t_\lambda\, C = C$ or $t^n_\lambda\, C = 0$ for some positive integer n. □

Theorem 3.32. *Let $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2$ and $A = A_1 \times A_2$ is $\mathfrak{S}$-graded ring where A_i is $\mathfrak{S}_i$-graded ring with $rank(\mathfrak{S}_i) = 2$, for $i = 1, 2$. Let $D = D_1 \times D_2$ be a graded finitely generated comultiplication A-module. Suppose that $C = C_1 \times C_2$ be a graded submodule of D. Then the following are equivalent:*

(a) *C is gr-2^{abs}_{st}-secondary submodule of D.*

(b) *Either $C_1 = 0$ and C_2 is gr-2^{abs}_{st}-secondary submodule of D_2 or $C_2 = 0$ and C_1 is gr-2^{abs}_{st}-secondary submodule of D_1 or C_1, C_2 are gr-secondary submodule of D_1, D_2, respectively.*

Proof. (a) $\Rightarrow$ (*b*): Let $C = C_1 \times C_2$ be gr-2^{abs}_{st}-secondary submodule of D. Then $Ann_A(C) = Ann_{A_1}(C_1) \times Ann_{A_2}(C_2)$ is gr-2^{abs}-primary ideal of A by Theorem 3.20. Using (Theorem 2.26, [16]), we have $Ann_{A_1}(C_1) = A_1$ and $Ann_{A_2}(C_2)$ is a gr-2^{abs}-primary ideal of A_2 or $Ann_{A_2}(C_2) = A_2$ and

$Ann_{A_1}(C_1)$ is a gr-2^{abs}-primary ideal of A_1 or $Ann_{A_1}(C_1)$ and $Ann_{A_2}(C_2)$ are gr-primary ideals of A_1 and A_2, respectively. Assume that $Ann_{A_1}(C_1) = A_1$ and $Ann_{A_2}(C_2)$ is a gr-2^{abs}-primary ideal of A_2, then $C_1 = 0$ and C_2 is gr-2^{abs}_{st}-secondary submodule of D_2 by Theorem 3.9. Similarly, if $Ann_{A_2}(C_2) = A_2$ and $Ann_{A_1}(C_1)$ is a gr-2^{abs}-primary ideal of A_1, then $C_2 = 0$ and C_1 is gr-2^{abs}_{st}-secondary submodule of D_1. If the last case hold, then since D_1 (resp. D_2) is a gr-com A_1 (resp. A_2) module, C_1 (resp. C_2) is gr-secondary submodule of D_1 (resp. D_2) by Lemma 3.31.

(b) $\Rightarrow$ (a): By Theorem 3.30. □

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