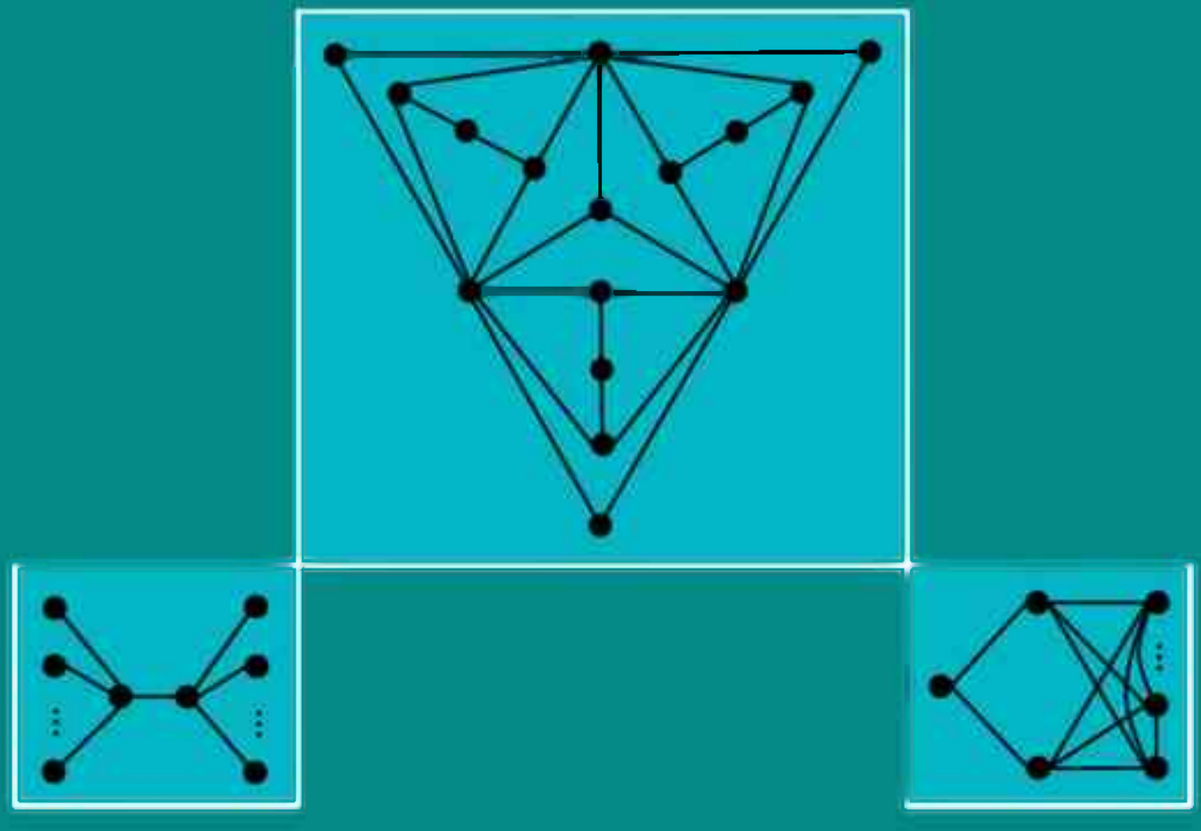
Radu Buzatu

Covering Graphs by Convex Sets



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Department of Mathematics

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Covering Graphs by Convex Sets

Monograph

Approved by
the Scientific Council of
the Moldova State University

Chisinau 2021

519.1
B 98

Recommended by the Department of Mathematics of the Moldova State University.

Mathematics Subject Classification (2010): 05A18, 05C35, 52A30, 68Q25.

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This monograph is published with the support of the scientific project "Deterministic and stochastic methods for solving optimization and control problems" (20.80009.5007.13).

CIP description of the National Book Chamber

Buzatu, Radu.

Covering Graphs by Convex Sets: Monograph / Radu Buzatu; Moldova State University, Department of Mathematics. – Chisinau: EPC MSU, 2021. – 106 p.: fig., tab.
Includes bibliog. refs.: p. 101-102 (26 titles).
Includes index: p. 103-106.
70 copies.

ISBN 978-9975-158-74-9.
519.1
B 98

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ISBN 978-9975-158-74-9

Preface

This monograph is intended to present the main results obtained by the author related to the convex covering and partitioning of graphs. It has been designed to be useful for research workers in the field, graduate students, and advanced undergraduates.

A *convexity* $\mathcal{C}$ in a graph $G = (X, U)$ is a family of subsets of X such that $\mathcal{C}$ is closed under intersection and contains both X and $\emptyset$. The elements of $\mathcal{C}$ are called *convex sets*. In this book, we consider the *geodetic convexity* $\mathcal{C}$ in a graph G, when each set S in $\mathcal{C}$ contains all vertices belonging to any shortest path between two vertices of S. This convexity is also known as *d-convexity*.

The problem of covering a graph by convex sets is a variation of the classical set covering problem that is known especially due to its essential role in developing the theory of complexity of algorithms, solving discrete optimization problems, and multiple practical applications in the engineering field. The problem of dividing a graph into convex sets is a particular case of covering a graph by convex sets, provided that all sets are pairwise disjoint. A comprehensive analysis of these two problems constitutes the core of the monograph.

The book is organized in seven chapters. Chapters 1 introduces basic notions of graph theory and defines convex covers and partitions of graphs. Chapter 2 presents several properties of convex covers of graphs. The NP-completeness of convex covering problems of graphs is shown in Chapter 3. Chapters 4 deals with special types of convex 2-covers of graphs. Chapter 5 focuses on nontrivial convex covers and partitions of trees. Several findings concerning convex covers of graphs resulted from

some graph operations are presented in Chapter 6. The last chapter of the monograph, Chapter 7, provides details on the use of binary linear programming approach to solve convex covering problems of graphs. The book also contains a bibliography, an index of terms, and an index of symbols.

Chisinau, Republic of Moldova Radu Buzatu

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Chapter 1
Basic Notions

By a *graph* we mean an ordered pair $G = (X, U)$ consisting of a finite nonempty set X of *vertices* and a finite set U of *edges*, where each edge is an unordered pair of vertices. The vertex set X and the edge set U of a graph G can be denoted by $X(G)$ and $U(G)$, respectively. A graph is *finite* if both its vertex set and edge set are finite. In this book, we consider only finite graphs. The *order* of G is the cardinality of $X(G)$, usually denoted by n. The *size* of G is the cardinality of $U(G)$, denoted by m. The graph with no vertices (and hence no edges) is the *null graph*.

An edge between two vertices x and y is denoted by (x, y) or simply by xy. Two vertices x and y are said to be *adjacent* or *neighbors* if there is an edge between them, and we denote this fact by writing $x \sim y$. An edge $e = (x, y)$ is said to be *incident* to vertices x and y, and vice versa. The *neighborhood* $\Gamma(x)$ of a vertex $x \in X$ is the set of neighbors of x in $G = (X, U)$. The *degree* of a vertex $x \in X$, denoted $deg(x)$, is the number of neighbors of x, i.e., $deg(x) = |\Gamma(x)|$. Vertices of degree 0 and 1 are called *isolated vertices* and *terminal vertices*, respectively. A set $S \subseteq X$ is a *clique* if every pair of vertices of S is adjacent in G. If $\Gamma(x)$ is a clique for a vertex $x \in X$, then x is called *simplicial*. Also, a vertex x is called *universal* if $\Gamma(x) = X \backslash \{x\}$.

By a *subgraph* H of a graph G we mean a graph H such that $X(H) \subseteq X(G)$ and $U(H) \subseteq U(G)$. We say that H is an *induced subgraph* of G if all the edges between the vertices in $X(H)$ from $U(G)$ are in $U(H)$. If H is an induced subgraph of a graph G and $S = X(H)$, then we say that H is the subgraph *induced by* set S in G and we write $H = G[S]$.

A *path* in a graph $G = (X, U)$ is a sequence of distinct vertices $[x_1, x_2, \dots, x_k]$ such that $(x_i, x_{i+1}) \in U$ for $i = \overline{1, k-1}$. If for such a path with $k \geq 3$, (x_1, x_k) is also an edge of G, then $[x_1, x_2, \dots, x_k]$ is a *cycle*. A cycle is called *simple* if it has no repeated vertices. The length of a path or a cycle is the number of edges in it. A graph consisting of a single path is called a *path graph*. A graph having no cycle is *acyclic*. A graph consisting of a single cycle is called a *cycle graph*. A cycle graph of order n is denoted by C_n. In particular, C_3 is often referred to as a *triangle*.

A graph $G = (X, U)$ is said to be *connected* if for all pairs of vertices $x, y \in X$, there is a path in G from x to y; otherwise, it is said to be *disconnected*. A *connected component* of G is a connected subgraph that is maximal by inclusion. A *cut-vertex* is a vertex whose removal produces a graph with more connected components than in the original graph. A *tree* is a connected acyclic graph.

A *complete graph*, denoted by K_n, is a graph of order n in which any two distinct vertices are adjacent. A *bipartite graph* $G = (X, U)$ is a graph whose vertex set X can be partitioned into two nonempty subsets X_1 and X_2 such that each edge of U connects one vertex of X_1 to one vertex of X_2. If every vertex of X_1 is adjacent to every vertex of X_2, then we say that G is a *complete bipartite graph*, and write $G = K_{p,q}$, where $p = |X_1|$ and $q = |X_2|$. A complete bipartite graph of the form $K_{1,q}$ is called a *star*. A graph $G = (X, U)$ is *tripartite* if X can be partitioned into three subsets X_1, X_2, X_3 such that each edge of U connects a vertex of X_i to a vertex of X_j, $i \neq j$, where $i, j \in \{1,2,3\}$.

We say that a set $S \subseteq X$ of a graph $G = (X, U)$ is *nontrivial* if $3 \leq |S| \leq |X| - 1$. At the same time, if $1 \leq |S| \leq 2$, then S is a *trivial set*. An *independent set* of a graph G is a subset of X such that no two vertices in X are adjacent. The *independence number* $\alpha(G)$ of a graph G is the cardinality of the largest independent set. A subset E of U is called an *edge cover* of G if, for each vertex $x \in X$, there is an edge $e \in E$ incident to x. The *edge cover number* $\beta(G)$ of a graph G is the size of the smallest edge cover of G. A *dominating set* of a graph G is a subset D of $X(G)$ such that every vertex not in D is adjacent to at least one member of D. A *minimum dominating set* is a dominating set of the smallest size in a given graph. The *domination number* $\gamma(G)$ is the number of vertices in a minimum

dominating set of G.

Let $G = (X, U)$ be a graph. The *distance* $d(x, y)$ between two vertices $x, y \in X$ is the length of the shortest path between x and y. The *diameter* of G, denoted by $diam(G)$, is the distance between two farthest vertices of G. If there is no path between two vertices $x, y \in X$, then we consider that $d(x, y) = +\infty$.

The distance function $d: X \times X \to \mathbb{N}$ associated to a graph $G = (X, U)$ satisfies, for every $x, y, z \in X$, the following properties:

1. $d(x, y) \geq 0$, equality holds if and only if $x = y$;

2. $d(x, y) = d(y, x)$;

3. $d(x, y) \leq d(x, z) + d(z, y)$.

Hence, for every graph $G = (X, U)$, d is a *metric* on X and the pair (X, d) is a *metric space*. The set

$$\langle x, y \rangle = \{z \in X: d(x, z) + d(z, y) = d(x, y)\}$$

is called *metric segment* between vertices x and y [1]. If G is a disconnected graph and vertices x and y belong to different connected components of G, then we consider that $\langle x, y \rangle = \emptyset$.

Definition 1.1. *A set $S \subseteq X$ is **convex** in a graph $G = (X, U)$ if $\langle x, y \rangle \subseteq S$ for any two vertices $x, y \in S$.*

Definition 1.2. *The **convex hull** of $S \subseteq X$, denoted $d - conv(S)$, is the smallest convex set containing S.*

Definition 1.3. *A family of sets is called a **convex cover** of a graph $G = (X, U)$ and is denoted by $\mathcal{P}(G)$ if the following conditions hold:*

a) every set of $\mathcal{P}(G)$ is convex in G;

b) $X = \bigcup_{S \in \mathcal{P}(G)} S$;

c) $S \not\subseteq \bigcup_{\substack{C \in \mathcal{P}(G) \\ C \neq S}} C$ for every $S \in \mathcal{P}(G)$.

If $|\mathcal{P}(G)| = p$, then this family is called a *convex p-cover* of G and is denoted by $\mathcal{P}_p(G)$. Initially, the concept of *convex p-cover* of a graph was defined and studied by Artigas et al. in [2,3]. In particular, a family $\mathcal{P}(G)$ is called a *convex partition* of G if it is a convex cover of G and any two

sets of $\mathcal{P}(G)$ are disjoint. A convex p-cover is called a *convex p-partition* if it is a convex partition of a graph. Clearly, every graph G has a convex n-partition, where $n = |X(G)|$. If $\mathcal{P}_p(G)$ is a convex p-partition and each set of $\mathcal{P}_p(G)$ is a clique, then $\mathcal{P}_p(G)$ is called a *clique p-partition* of G.

Definition 1.4. *The family $\mathcal{P}(G)$ is called the **nontrivial convex cover** of a graph $G = (X, U)$ if every set $S \in \mathcal{P}(G)$ satisfies the inequalities $3 \leq |S| \leq |X| - 1$. Consequently, the elements of $\mathcal{P}(G)$ are called **nontrivial convex sets**.*

If a nontrivial convex cover $\mathcal{P}(G)$ is a convex partition, we say that $\mathcal{P}(G)$ is a *nontrivial convex partition*. *Nontrivial convex p-cover and nontrivial convex p-partition* of G are defined analogically.

Indeed, there are graphs for which there are no nontrivial convex covers. For instance, every convex simple graph has no nontrivial convex covers. Note that a graph is called *convex simple* if it does not contain a nontrivial convex set [8].

Definition 1.5. *The **minimum convex cover number** $\varphi_c^{min}(G)$ of a graph G is the least integer $p \geq 2$ for which G has a convex p-cover. Similarly, the **minimum convex partition number** $\theta_c^{min}(G)$ of G is the least integer $p \geq 2$ for which G has a convex p-partition.*

By analogy, the following numbers can be defined:

$\varphi_{cn}^{min}(G)$ – *minimum nontrivial convex cover number of* G;

$\theta_{cn}^{min}(G)$ – *minimum nontrivial convex partition number of* G;

$\varphi_{cn}^{max}(G)$ – *maximum nontrivial convex cover number of* G;

$\theta_{cn}^{max}(G)$ – *maximum nontrivial convex partition number of* G.

The respective families of convex sets are denoted by:

$$\mathcal{P}_{\varphi_c^{min}}(G), \mathcal{P}_{\theta_c^{min}}(G), \mathcal{P}_{\varphi_{cn}^{min}}(G), \mathcal{P}_{\theta_{cn}^{min}}(G), \mathcal{P}_{\varphi_{cn}^{max}}(G), \mathcal{P}_{\theta_{cn}^{max}}(G).$$

It is obvious that for any graph G, we have $\varphi_c^{min}(G) \leq \theta_c^{min}(G)$. If G admits a partition into nontrivial convex sets, then $\theta_c^{min}(G) \leq \theta_{cn}^{min}(G)$ and also:

$$\varphi_{cn}^{min}(G) \leq \theta_{cn}^{min}(G) \leq \theta_{cn}^{max}(G) \leq \varphi_{cn}^{max}(G).$$

Anyway, if G can be covered by nontrivial convex sets, then it is clear that $\varphi_c^{min}(G) \leq \varphi_{cn}^{min}$.

Definition 1.6. *A vertex $x \in X$ is called* ***resident*** *in $\mathcal{P}(G)$ if x belongs to only one set of $\mathcal{P}(G)$.*

By definition, each set of $\mathcal{P}(G)$ contains at least one resident vertex in $\mathcal{P}(G)$. If $\mathcal{P}(G)$ is a convex partition of G, then all vertices of every set of $\mathcal{P}(G)$ are resident in $\mathcal{P}(G)$.

For additional details and information on basic graph theory and convexity in graphs, we refer the reader to [5,6].

All the authors' results exposed in this book were published in [7-14].

Chapter 2
Basic Properties

In this chapter, some properties of minimum convex covers and minimum convex partitions of graphs are established. In addition, several theorems regarding the existence of connected graphs with given parameters are proved.

2.1 Minimum convex covers

Theorem 2.1. *If* $\varphi_c^{min}(G) \geq 3$, *then for any two sets* $A, B \in \boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G)$, $A \neq B$, *there exists a set* $C \in \boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G) \backslash \{A, B\}$ *such that there exist vertices* $a \in A$, $b \in B$ *and* $c \in C \backslash (A \cup B)$, *where* $c \in \langle a, b \rangle$.

Proof. Assume the converse. Suppose there exist sets $A, B \in \boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G)$, $A \neq B$, such that for all vertices $a \in A$ and $b \in B$, $\langle a, b \rangle \subseteq A \cup B$. Thus, since $d - conv(A \cup B) = A \cup B$, there is convex cover $\boldsymbol{\mathcal{P}}(G)$ which consists of all sets of $\boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G)$, except for A and B, and contains an additional set $S = A \cup B$. Finally, we get a contradiction because $|\boldsymbol{\mathcal{P}}(G)| = \left|\boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G)\right| - 1$. □

Theorem 2.2. *If* $\varphi_c^{min}(G) \geq 3$, *then for each set* $A \in \boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G)$, *there exist sets* $B, C \in \boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G) \backslash \{A\}$, $B \neq C$, *such that there exist vertices* $a \in A \backslash (B \cup C)$, $b \in B$ *and* $c \in C$, *where* $a \in \langle b, c \rangle$.

Proof. Assume the converse. Suppose that there exists a set $A \in \boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G)$ such that for any two sets $B, C \in \boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G) \backslash \{A\}$, $B \neq C$, the equality

$A \cap (\langle b,c\rangle \backslash (B \cup C)) = \emptyset$ holds for all vertices $b \in B$, $c \in C$. This yields that $d - conv\left(\bigcup_{S \in \mathcal{P}_{\varphi_c^{min}(G)\backslash\{A\}}} S\right) = \bigcup_{S \in \mathcal{P}_{\varphi_c^{min}(G)\backslash\{A\}}} S$. Therefore, we obtain a convex 2-cover

$$\mathcal{P}_2(G) = \mathcal{P}_{\varphi_c^{min}}(G) = \left\{\bigcup_{S \in \mathcal{P}_{\varphi_c^{min}(G)\backslash\{A\}}} S, A\right\}.$$

So, $\varphi_c^{min}(G) = 2$, and this contradicts the condition of the theorem that $\varphi_c^{min}(G) \geq 3$. □

Considering nontrivial convex cover as a particular case of a convex cover, Theorems 2.1 and 2.2 have the following consequences.

Corollary 2.1. *If $\varphi_{cn}^{min}(G) \geq 3$, then for any two sets $A, B \in \mathcal{P}_{\varphi_{cn}^{min}(G)}$, $A \neq B$, there exists a set $C \in \mathcal{P}_{\varphi_{cn}^{min}}(G)\backslash\{A, B\}$ such that there exist vertices $a \in A$, $b \in B$ and $c \in C\backslash(A \cup B)$, where $c \in \langle a, b\rangle$.*

Corollary 2.2. *If $\varphi_{cn}^{min}(G) \geq 3$, then for each set $A \in \mathcal{P}_{\varphi_{cn}^{min}}(G)$, there exist sets $B, C \in \mathcal{P}_{\varphi_{cn}^{min}}(G)\backslash\{A\}$, $B \neq C$, such that there exist vertices $a \in A\backslash(B \cup C)$, $b \in B$ and $c \in C$, where $a \in \langle b, c\rangle$.*

Theorem 2.3. *Let $G = (X, U)$ be a connected graph and let S be a family of subsets of X with properties:*

a) $|S| \geq 2$;

b) each $Y \in S$ is a clique;

c) $X\backslash \bigcup_{Y \in S} Y$ is not a clique;

d) $Y \cap Z = \emptyset$ for all $Y, Z \in S$;

e) for each set $Y \in S$, the equality $\Gamma(y) = (Y\backslash\{y\}) \cup (X\backslash \bigcup_{Z \in S} Z)$ holds for every vertex $y \in Y$.

Then the following conditions hold:

a) $\varphi_c^{min}(G) \geq \alpha(G)$, $\theta_c^{min}(G) \geq \alpha(G)$;

b) $\varphi_{cn}^{min}(G) \geq \alpha(G)$ if G can be covered by nontrivial convex sets;

c) $\theta_{cn}^{min}(G) \geq \alpha(G)$ if G can be partitioned into nontrivial convex sets;

d) every convex set of G is a clique.

Proof. Consider two nonadjacent vertices a, b of $X \setminus \bigcup_{Y \in S} Y$ and two vertices y, z such that $y \in Y$, $z \in Z$, where $Y, Z \in S$, $Y \neq Z$. Note that vertices y and z are nonadjacent by definition.

By property e), we have $\bigcup_{Y \in S} Y \subseteq \langle a, b \rangle$, $X \setminus \bigcup_{Y \in S} Y \subseteq \langle y, z \rangle$ and thus $X \setminus \bigcup_{Y \in S} Y \subseteq d - conv(\bigcup_{Y \in S} Y)$, $\bigcup_{Y \in S} Y \subseteq d - conv(X \setminus \bigcup_{Y \in S} Y)$. Furthermore, we have

$$d - conv(\bigcup_{Y \in S} Y) = d - conv(X \setminus \bigcup_{Y \in S} Y) = X.$$

Thus, there is no proper convex subset of X containing vertices a, b or y, z. This means that every convex set of G is a clique.

Let $M \subseteq X$ be the maximum independent set of G. In addition, by property e) it follows that $M \subseteq X \setminus \bigcup_{Y \in S} Y$, or $M \subseteq \bigcup_{Y \in S} Y$ such that each element of M belongs to exactly one set of S. Further, every convex cover of G has at least $|M| = \alpha(G)$ sets. This implies the inequalities:

$$\varphi_c^{min}(G) \geq \alpha(G), \theta_c^{min}(G) \geq \alpha(G).$$

Moreover, if a graph G can be covered by nontrivial convex sets, then $\varphi_{cn}^{min}(G) \geq \alpha(G)$. Also, if G can be partitioned into nontrivial convex sets, then $\theta_{cn}^{min}(G) \geq \alpha(G)$. □

2.2 The existence of graphs with given parameters

For all $n \in N$, $n \geq 2$, there exists a connected graph on n vertices that has a convex 2-cover and a convex 2-partition. An example of such a graph is a path graph of order $n \geq 2$. In addition, for every graph that has a nontrivial convex 2-cover, the inequality $n \geq 4$ holds because every set belonging to a convex 2-cover is nontrivial and contains at least one resident vertex. On the other hand, for every graph that has a nontrivial convex 2-partition, the inequality $n \geq 6$ is satisfied because its sets are nontrivial and disjoint.

It is clear that for any graph G on n vertices, where $n = 2$ or $n = 3$, we have $\varphi_c^{min}(G) = \theta_c^{min}(G) = 2$. Now we prove theorems regarding the existence of connected graphs with fixed values of $\varphi_c^{min}(G)$, $\theta_c^{min}(G)$, $\varphi_{cn}^{min}(G)$, $\theta_{cn}^{min}(G)$, $\varphi_{cn}^{max}(G)$, $\theta_{cn}^{max}(G)$.

Theorem 2.4. *If $G = (X, U)$ is a connected graph on $n \geq 4$ vertices, then $\theta_c^{min}(G) \leq \beta(G)$.*

Proof. Let E be a minimum edge cover of graph G and let F be a family of two-vertex sets corresponding to edges of E. Since every set of F is convex in G and $X = \bigcup_{S \in F} S$, it follows that F is a convex cover of G. Moreover, if there are two sets $A, B \in F$, which contain the same vertex c, then we remove A from F and add the set $A \backslash \{c\}$ to F. By repeating this procedure for any two sets of F, which have a common vertex, we transform F into a convex partition of graph G for wich $|F| = \beta(G)$. Therefore, we have $\theta_c^{min}(G) \leq \beta(G)$. □

Theorem 2.5. *For any two numbers $p, n \in N$, $2 \leq p \leq n-2$, there exists a connected graph G on n vertices such that $\varphi_c^{min}(G) = p$.*

Proof. If $p = 2$, then G is a path graph of order n for which $\varphi_c^{min}(G) = 2$.

If $p \geq 3$, we construct a graph $G = (X, U)$ as follows:

Step 1. We define a set of vertices $X_1 = \{x_1, x_2, \dots, x_p\}$, where any two vertices of X_1 are nonadjacent, i.e., X_1 is an independent set.

Step 2. If $p < n-2$, then we define a set of vertices $X_2 = X_1 \cup Z$, where $Z = \{z_1, z_2, \dots, z_{n-p-2}\}$ and $Z \cup \{x_1\}$ is a clique. Otherwise, X_2 is equal to X_1 and $Z = \emptyset$.

Step 3. $X = X_2 \cup \{y_1, y_2\}$, where $\Gamma(y_1) = \Gamma(y_2) = X_2$.

The resulted graph G is exhibited in Figure 2.1. Clearly, $|X| = n$.

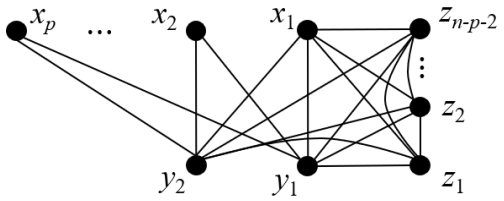


Figure 2.1. *Graph G on n vertices such that $\theta_c^{min}(G) = \varphi_c^{min}(G) = p$, $3 \leq p \leq n-2$.*

Since X_1 is a maximum independent set in G, $\alpha(G) = |X_1| = p$. The family of sets $\{\{y_1\}, \{y_2\}\}$ satisfies the conditions of Theorem 2.3. Thus, $\varphi_c^{min}(G) \geq p$ and every convex set of G is a clique.

It remains to show that there exists a convex p-cover of G. Graph G has a convex p-cover $\mathcal{P}_p(G)$ that consists of cliques $\{x_1, y_1\} \cup Z$, $\{x_2, y_2\}$,

$\{x_3\}$, $\{x_4\}$, … , $\{x_p\}$. Since $\varphi_c^{min}(G) \geq p$, it follows that $\mathcal{P}_p(G)$ is the minimum convex cover of G and $\varphi_c^{min}(G) = p$ □

In view of proof of Theorem 2.5, we obtain Corollary 2.3.

Corollary 2.3. *For any two numbers $p, n \in N$, $2 \leq p \leq n-2$, there exists a connected graph G on n vertices such that $\theta_c^{min}(G) = p$.*

Let us prove several theorems regarding the existence of graphs with fixed values of $\varphi_{cn}^{min}(G)$ and $\theta_{cn}^{min}(G)$.

Theorem 2.6. *For any two numbers $p, n \in N$, $2 \leq p \leq \left\lfloor \frac{n}{3} \right\rfloor$, there exists a connected graph G on n vertices such that $\theta_{cn}^{min}(G) = p$.*

Proof. We construct a graph $G = (X, U)$ as follows:

Step 1. We define a set $X_1 = \{x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, \dots, x_{p,1}, x_{p,2}\}$, where $x_{i,1} \sim x_{i,2}$ for each i, $1 \leq i \leq p$.

Step 2. If $3p < n$, then we define a set of vertices $X_2 = X_1 \cup Z$, where $Z = \{z_1, z_2, \dots, z_{n-3p}\}$ such that $Z \cup \{x_{1,1}, x_{1,2}\}$ is a clique. Otherwise, $X_2 = X_1$ and $Z = \emptyset$.

Step 3. $X = X_2 \cup Y$, where $Y = \{y_1, y_2, \dots, y_p\}$ such that $\Gamma(y_i) = X_2$ for each i, $1 \leq i \leq p$.

The resulted graph G is represented in Figure 2.2. Clearly, $|X| = n$.

Since Y is a maximum independent set in G, $\alpha(G) = |Y| = p$. The family $\{\{y_1\}, \{y_2\}, \dots, \{y_p\}\}$ satisfies the conditions of Theorem 2.3. Consequently, if G can be partitioned into nontrivial convex sets, then $\theta_{cn}^{min}(G) \geq p$. Also, every convex set of G is a clique.

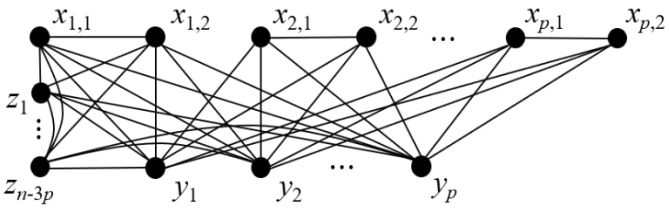


Figure 2.2. *Graph G on n vertices such that $\theta_{cn}^{min}(G) = p$, $2 \leq p \leq \left\lfloor \frac{n}{3} \right\rfloor$.*

It remains to show that there exists a nontrivial convex p-partition of G. Graph G has a nontrivial convex p-partition $\mathcal{P}_p(G)$ that consist of the following cliques: $\{x_{1,1}, x_{1,2}, y_1\} \cup Z$, $\{x_{2,1}, x_{2,2}, y_2\}$, $\{x_{3,1}, x_{3,2}, y_3\}$, … , $\{x_{p,1}, x_{p,2}, y_p\}$. So, since $\theta_{cn}^{min}(G) \geq p$, it follows that $\mathcal{P}_p(G)$ is the minimum nontrivial convex partition of G and $\theta_{cn}^{min}(G) = p$. □

By C_4 we denote a cycle graph on 4 vertices.

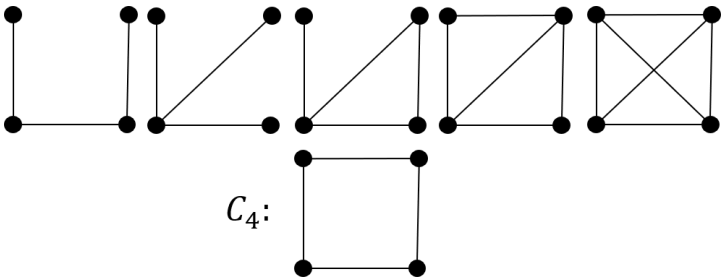


Figure 2.3. *All connected graphs on* 4 *vertices.*

Theorem 2.7. *If G is a connected graph on* 4 *vertices, then $\varphi_{cn}^{min}(G) = 2$ if and only if $G \neq C_4$.*

Proof. By the definition of nontrivial convex cover, a graph G on 4 vertices has a nontrivial convex p-cover if and only if $p = 2$. In Figure 2.3, are represented all connected graphs on 4 vertices. It can be easily checked that every graph presented in Figure 2.3, except for the cycle graph C_4, has a nontrivial convex 2-cover. Now, if we recall that the minimum nontrivial convex cover number is the least integer $p \geq 2$ for which graph G has a nontrivial convex p-cover, we get $\varphi_{cn}^{min}(G) = 2$ for every connected graph G on 4 vertices, where $G \neq C_4$. □

Theorem 2.8. *If a connected graph G on $n \geq 5$ vertices can be covered by nontrivial convex sets, then $\varphi_{cn}^{min}(G) \leq n - 3$.*

Proof. There is no nontrivial convex n-cover or a nontrivial convex $(n-1)$-cover of G, because every convex set of a nontrivial convex cover has at least one resident vertex in this cover and contains more than two vertices.

Let us prove that for every graph G, the inequality $\varphi_{cn}^{min}(G) \leq n - 3$ holds. The proof is by reductio ad absurdum. Assume that there exists a

graph $G = (X, U)$ on $n \geq 5$ vertices such that $\varphi_{cn}^{min}(G) = n - 2$. Consequently, we have $\varphi_{cn}^{min}(G) \geq 3$. Let $\mathcal{P}_{\varphi_{cn}^{min}}(G)$ be a minimum nontrivial convex cover of G. In this case, every set $S \in \mathcal{P}_{\varphi_{cn}^{min}}(G)$ satisfies the equality $|S| = 3$ and contains exactly one resident vertex in $\mathcal{P}_{\varphi_{cn}^{min}}(G)$. Further, there are two vertices $x, y \in X$, which are common for all sets of $\mathcal{P}_{\varphi_{cn}^{min}}(G)$. Notice that $x \sim y$, otherwise the connectivity of nontrivial convex sets of $\mathcal{P}_{\varphi_{cn}^{min}}(G)$ implies that $\Gamma(x) = \Gamma(y) = X \backslash \{x, y\}$ and further $d - conv(\{x, y\}) = X$. According to Corollary 2.1, all vertices of $X \backslash \{x, y\}$ are nonadjacent in G. Therefore, we get a nontrivial convex 2-cover $\mathcal{P}_2(G) = \mathcal{P}_{\varphi_{cn}^{min}}(G) = \{\{x, y, z\},\ X \backslash \{z\}\}$, where $z \in X \backslash \{x, y\}$. Finally, $\varphi_{cn}^{min}(G) = 2$. This contradiction concludes the proof of the theorem. □

Theorem 2.9. *For any two numbers $p, n \in N$, $2 \leq p \leq n - 3$, there exists a connected graph G on n vertices such that $\varphi_{cn}^{min}(G) = p$.*

Proof. We construct a graph $G = (X, U)$ as follows:

Step 1. We define a set of vertices $X_1 = \{x_1, x_2, \dots, x_p\}$, where any two vertices of X_1 are nonadjacent, i.e., X_1 is an independent set.

Step 2. If $p < n - 3$, then we define a set of vertices $X_2 = X_1 \cup Z$, where $Z = \{z_1, z_2, \dots, z_{n-p-3}\}$ such that $Z \cup \{x\}$ is a clique for each $x \in X_1$. Otherwise, $X_2 = X_1$ and $Z = \emptyset$.

Step 3. $X = X_2 \cup Y$, where $Y = \{y_1, y_2, y_3\}$ and $\Gamma(y_1) = X_1 \cup \{y_2\}$, $\Gamma(y_2) = X_2 \cup \{y_1, y_3\}$, $\Gamma(y_3) = X_2 \cup \{y_2\}$.

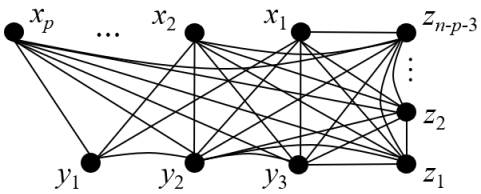


Figure 2.4. *Graph G on n vertices such that $\varphi_{cn}^{min}(G) = p$, $2 \leq p \leq n - 3$.*

The resulted graph is represented in Figure 2.4. Clearly, $|X| = n$.

Since X_1 is a maximum independent set in G, $\alpha(G) = |X_1| = p$. The family $\{\{x_1\}, \{x_2\}, \dots, \{x_p\}\}$ satisfies the conditions of Theorem 2.3. Thus,

if G can be covered by nontrivial convex sets, then $\varphi_{cn}^{min}(G) \geq p$. Also, every convex set of G is a clique.

It remains to show that there exists a nontrivial convex p-cover of G. Graph G has a nontrivial convex p-cover $\mathcal{P}_p(G)$ that consists of the following cliques: $\{x_1, y_2, y_3\} \cup Z$, $\{x_2, y_1, y_2\}$, $\{x_3, y_1, y_2\}$, … , $\{x_p, y_1, y_2\}$. Since $\varphi_{cn}^{min}(G) \geq p$, $\mathcal{P}_p(G)$ is the minimum nontrivial convex cover of G and $\varphi_{cn}^{min}(G) = p$. □

Theorem 2.10. *For any two numbers $p, n \in N$, $2 \leq p \leq n-2$, there exists a connected graph G on n vertices such that $\varphi_{cn}^{max}(G) = p$.*

Proof. We construct a graph $G = (X, U)$ as follows:

Step 1. If $n - p = 2$, we take a path graph $G' = [x, y, z]$. Otherwise, we take a convex simple graph $G' = (X', U')$ with $n - p + 1$ vertices, and chose two adjacent vertices x, y of X'.

Step 2. We define a set of vertices $X_1 = \{x_1, x_2, \dots, x_{p-1}\}$ and get $X = X' \cup X_1$ such that $x \sim x_i$ for each i, $1 \leq i \leq p-1$.

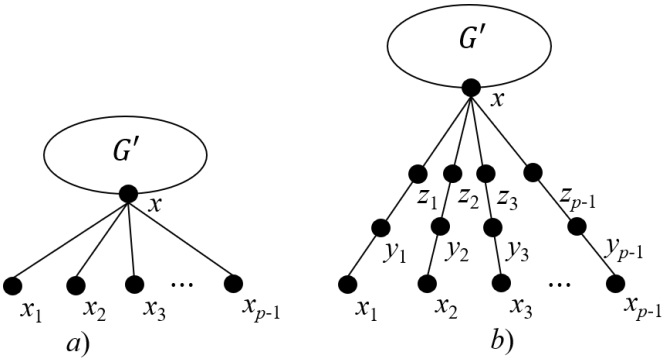


Figure 2.5. *Graph G on n vertices such that $\varphi_{cn}^{max}(G) = p$, $2 \leq p \leq n-2$ (case a)), and graph G on n vertices such that $\theta_{cn}^{max}(G) = p$, $2 \leq p \leq \left\lfloor \frac{n}{3} \right\rfloor$ (case b).*

The resulted graph is represented in Figure 2.5 a). It is easy to see that $|X| = n$.

If G' is a path graph, then the theorem is trivial. We consider that G' is a convex simple graph and $|X'| \geq 4$. Assume that x is adjacent to all vertices of $X' \backslash \{x\}$. Since a star graph with at least 4 vertices is not convex simple, there are two adjacent vertices $u, v \in X' \backslash \{x\}$. Therefore, we see

that $\{x, u, v\}$ is a nontrivial convex set in G', which runs counter to the fact that graph G' is convex simple. Thus, exists a vertex $z \in X' \backslash \{x\}$ such that $d(x, z) \geq 2$. By construction of G, a minimal nontrivial convex set containing z is X'. Graph G has a nontrivial convex p-cover $\mathcal{P}_p(G)$ that consist of the following sets: X', $\{x_1, x, y\}$, $\{x_2, x, y\}$, … , $\{x_{p-1}, x, y\}$. Each vertex of $X_1 \cup \{z\}$ is resident in $\mathcal{P}_p(G)$. Consequently, $\mathcal{P}_p(G)$ is a maximum nontrivial convex cover of G. □

Theorem 2.11. *For any two numbers $p, n \in N$, $2 \leq p \leq \left\lfloor \frac{n}{3} \right\rfloor$, there exists a connected graph G on n vertices such that $\theta_{cn}^{max}(G) = p$.*

Proof. We construct a graph $G = (X, U)$ as follows:

Step 1. If $n = 3p$, we take a path graph $G' = [x, y, z]$. Otherwise, we take a convex simple graph $G' = (X', U')$ with $n - 3p + 3$ vertices, and chose two adjacent vertices x, y of X'.

Step 2. We define $p - 1$ paths $L_i = \{x_i, y_i, z_i\}$ and get $X = X' \cup \bigcup_{i=1}^{p-1} L_i$ such that $x \sim x_i$ for each i, $1 \leq i \leq p - 1$.

The resulted graph is represented in Figure 2.5 b). It is easy to see that $|X| = n$.

If G' is a path graph, then the theorem is trivial. If G' is a convex simple graph, then considering the proof of Theorem 2.10, there exists a vertex $z \in X' \backslash \{x\}$ such that $d(x, z) \geq 2$, and a minimal nontrivial convex set containing z is X'. Hence, graph G has a nontrivial convex p-partition $\mathcal{P}_p(G)$ that consist of the following sets: X', $\{x_1, y_1, z_1\}$, $\{x_2, y_2, z_2\}$, … , $\{x_{p-1}, y_{p-1}, z_{p-1}\}$. Note that each vertex of $X' \cup \bigcup_{i=1}^{p-1} L_i$ is resident in $\mathcal{P}_p(G)$. So, $\mathcal{P}_p(G)$ is a maximum nontrivial convex partition of G. □

Chapter 3
NP-Completeness

Deciding whether a graph G has a convex p-cover, for a fixed $p \geq 2$, is said to be a convex p-cover problem. Similarly, deciding whether a graph G has a convex p-partition, for a fixed $p \geq 2$, is said to be a convex p-partition problem. By the analogy, we introduce nontrivial convex p-cover and nontrivial convex p-partition problems, where nontrivial convex sets are considered.

Theorem 3.1. [3] *The convex p-partition problem is NP-complete for a fixed $p \geq 2$.*

Theorem 3.2. [2] *The convex p-cover problem is NP-complete for a fixed $p \geq 3$.*

Deciding whether a graph has a convex 2-cover was declared as an open problem in [2].

For the sequel, it is important to mention the following result, which states that for a graph G, the convex hull of a subset $S \subseteq X(G)$ can be computed in polynomial time.

Theorem 3.3. [15] *Let $G = (X, U)$ be a graph. The convex hull of a subset $S \subseteq X$ can be computed in $O(|d - conv(S)|m)$ time.*

3.1 Two convex sets

Here, we prove that the convex 2-cover problem is NP-complete. We establish the NP-completeness of this problem by reduction from the well-known NP-complete 1-IN-3 3SAT problem [16].

Problem: 1-IN-3 3SAT.

Instance: Set $V = \{v_1, v_2, \dots, v_n\}$ of variables and collection of clauses $\mathcal{C} = \{c_1, c_2, \dots, c_m\}$ over V such that each clause $c \in \mathcal{C}$ has no negative literals and $|c| = 3$.

Question: Is there a truth assignment for V such that each clause in $\mathcal{C}$ has exactly one true literal?

We say that $\mathcal{C}$ is *satisfiable* if there exists a true assignment for V such that each clause in $\mathcal{C}$ has exactly one true variable.

Theorem 3.4. *The convex* 2*-cover problem is NP-complete.*

Proof. This problem is in NP because, by Theorem 3.3, verifying if a set is convex can be done in polynomial time. Also, it can be easily checked whether two convex sets cover a graph.

We reduce 1-IN-3 3SAT problem to convex 2-cover problem. First, we determine the structure of a particular graph $G = (X, U)$ for convex 2-cover problem from a generic instance $(V, \mathcal{C})$ of 1-IN-3 3SAT problem. Next, we prove that $\mathcal{C}$ is satisfiable if and only if G has a convex 2-cover. For this purpose, we prove that a convex 2-cover of G defines a truth assignment that satisfies $(V, \mathcal{C})$. Also, we prove that a truth assignment that satisfies $(V, \mathcal{C})$ defines a convex 2-cover of G.

The graph G is given by the set of vertices X together with the set of edges U. The vertex set X consists of the following sets:

a) $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$, $Y = \{f, y_1, y_2, y_3, y_4, y_5\}$,
$Z = \{t, z_1, z_2, z_3, z_4, z_5\}$;

b) $F = \{f_j : 1 \le j \le m\}$, $T = \{t_j : 1 \le j \le m\}$;

c) $L = \{l_j^i : 1 \le j \le m, 1 \le i \le 3\}$, $\mathcal{L} = \{\ell_j^i : 1 \le j \le m, 1 \le i \le 3\}$,
$Q = \{q_j^i : 1 \le j \le m, 1 \le i \le 3\}$.

We get $X = \mathcal{V} \cup Y \cup Z \cup F \cup T \cup L \cup Q \cup \mathcal{L}$. Every variable $v_i \in V$ corresponds to a vertex $v_i \in \mathcal{V}$. Every clause $c_j \in \mathcal{C}$ corresponds to eleven vertices: $f_j, t_j, l_j^1, l_j^2, l_j^3, \ell_j^1, \ell_j^2, \ell_j^3, q_j^1, q_j^2, q_j^3$.

The edge set U satisfies the following conditions:

a) $\mathcal{V} \cup Q$ is a clique in G;

b) $\Gamma(f) = \mathcal{V} \cup Q \cup F \cup \{y_3, y_4\}$ and $\Gamma(t) = \mathcal{V} \cup Q \cup T \cup \{z_3, z_4\}$;

c) $\Gamma(y_5) = F \cup \{y_3, y_4\}$ and $\Gamma(z_5) = T \cup \{z_3, z_4\}$;

d) there exist the following edges: (y_1, y_3), (y_1, y_4), (y_2, y_3), (y_2, y_4), (z_1, z_3), (z_1, z_4), (z_2, z_3), (z_2, z_4);

e) every clause $c_j = \{v_a, v_b, v_c\}$, $1 \le j \le m$, corresponds to eighteen edges: (l_j^1, u_a), (l_j^2, u_b), (l_j^3, u_c), (l_j^1, f_j), (l_j^2, f_j), (l_j^3, f_j), (ℓ_j^1, t_j), (ℓ_j^2, t_j), (ℓ_j^3, t_j), (q_j^1, ℓ_j^1), (q_j^2, ℓ_j^2), (q_j^3, ℓ_j^3), (l_j^1, ℓ_j^2), (l_j^1, ℓ_j^3), (l_j^2, ℓ_j^1), (l_j^2, ℓ_j^3), (l_j^3, ℓ_j^1), (l_j^3, ℓ_j^2).

It is easy to see that G can be constructed in polynomial time.

Without loss of generality, we skip the trivial case of 1-IN-3 3SAT problem, when $|\mathcal{C}| = 1$, and consider that $|\mathcal{C}| \ge 2$.

If $G = (X, U)$ has a convex 2-cover, then $\mathcal{C}$ is satisfiable.

Let $\mathcal{P}_2(G) = \{S_f, S_t\}$ be a convex 2-cover of G.

For every pair of elements $i, j \in \{1,2\}$, $d - conv(\{y_i, z_j\}) = X$. Let $y_1, y_2 \in S_f$ and $z_1, z_2 \in S_t$. We define two sets $S_1 = \{y_3, y_4, y_5, f\} \cup F$ and $S_2 = \{z_3, z_4, z_5, t\} \cup T$.

We have to enumerate some properties:

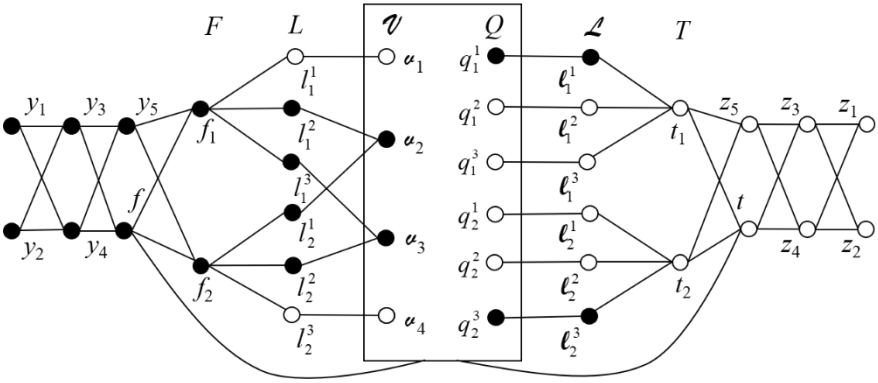


Figure 3.1. *The convex 2-cover of a graph G for the instance* $(V,\mathcal{C}) = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\})$. *A rectangle represents a clique.*

Property 1: *$S_1 \cap S_t = \emptyset$ and $S_2 \cap S_f = \emptyset$.*

Since $S_1 \subseteq d - conv(\{y_1, y_2\})$, $S_2 \subseteq d - conv(\{z_1, z_2\})$, we get $S_1 \subseteq S_f$, $S_2 \subseteq S_t$. Moreover, for each vertex $u \in F \cup \{f\}$, we have:

$$d - conv(\{u,t\} \cup T) \supseteq d - conv(\{f\} \cup F \cup S_2) \supseteq$$
$$\supseteq d - conv(S_1 \cup S_2) \supseteq d - conv\big(S_f \cup S_t\big) = X.$$

This implies that $u \notin S_t$ for each $u \in F \cup \{f\}$. Similarly, for each vertex $u \in T \cup \{t\}$, we have:

$$d - conv(\{u,f\} \cup F) \supseteq d - conv(\{t\} \cup T \cup S_1) \supseteq$$
$$\supseteq d - conv(S_1 \cup S_2) \supseteq d - conv\big(S_f \cup S_t\big) = X.$$

This entails that $u \notin S_f$ for each $u \in T \cup \{t\}$. Consequently, we obtain $S_1 \cap S_t = \emptyset$ and $S_2 \cap S_f = \emptyset$.

Property 2: *Sets $L, \mathcal{V}, Q, \mathcal{L}$ are uniquely interdependent*.

If vertex l_j^i belongs to S_t, then $\Gamma(l_j^i) \cap \mathcal{V} \subseteq S_t$ and ℓ_j^k belongs to S_t for each k, $1 \le k \le 3$, $k \ne i$.

If vertex ω_i belongs to S_t, then $\Gamma(\omega_i) \cap L \subseteq S_t$, and for each $l_j^a \in \Gamma(\omega_i) \cap L$, vertices ℓ_j^k belong to S_t for $1 \le k \le 3$, $k \ne a$.

Vertex ℓ_j^i belongs to S_f if and only if q_j^i belongs to S_f. If ℓ_j^i belongs to S_f, then $L' = \{l_j^k : 1 \le k \le 3, k \ne i\} \subseteq S_f$ and $\Gamma(l_j^k) \cap \mathcal{V}$ is contained in S_f for all $l_j^k \in L'$.

Property 3: *For each j, $1 \le j \le m$, exactly one vertex of $L_j = \{l_j^1, l_j^2, l_j^3\}$ belongs to S_t and exactly one vertex of $\mathcal{L}_j = \{\ell_j^1, \ell_j^2, \ell_j^3\}$ belongs to S_f*.

Exactly one vertex of each set $L_j = \{l_j^1, l_j^2, l_j^3\}$, $1 \le j \le m$, belongs to S_t. In the converse case, if two vertices $\{l_j^a, l_j^b\}$ of L_j belong to S_t, then $f_j \in S_t$. By Property 1, we get a contradiction. If there are no vertices of $L_j = \{l_j^1, l_j^2, l_j^3\}$ belonging to S_t, then $L_j \subset S_f$, $\mathcal{L}_j = \{\ell_j^1, \ell_j^2, \ell_j^3\} \subset S_f$ and $t_j \in S_f$. Now, again by Property 1, we have a contradiction.

For the same reason, exactly one vertex of each set $\mathcal{L}_j = \{\ell_j^1, \ell_j^2, \ell_j^3\}$, $1 \le j \le m$, belongs to S_f.

We associate $\mathcal{V}$ with V and L with $\mathcal{C}$ such that a convex 2-cover of G represents a truth assignment for V, where the variable v_i is true iff $\omega_i \in S_t$.

It follows from Properties 1, 2 and 3 that if graph G has a convex 2-cover $\mathcal{P}_2(G) = \{S_f, S_t\}$, then $\mathcal{C}$ is satisfiable. Let us remark that sets S_f, S_t are nontrivial and disjoint.

If $\mathscr{C}$ is satisfiable, then $G = (X, U)$ has a convex 2-cover.

Suppose that there exists a truth assignment that satisfies $(V, \mathscr{C})$. We construct a convex 2-cover $\mathscr{P}_2(G) = \{S_f, S_t\}$ as follows:

Step 1. Define the set $S_t = \{z_1, z_2, z_3, z_4, z_5, t\} \cup T$.

Step 2. For each true variable v_i of V, we add vertex u_i and the set $L' = \Gamma(u_i) \cap L$ to S_t, and for each $l_j^a \in L'$, we add vertices q_j^b, ℓ_j^b to S_t such that $\ell_j^b \sim l_j^a$ and $q_j^b \sim \ell_j^b$.

Step 3. Define the set $S_f = X \backslash S_t$.

Obviously, for the resulting convex 2-cover $\mathscr{P}_2(G) = \{S_f, S_t\}$, Properties 1, 2, and 3 are satisfied. Hence, if $\mathscr{C}$ is satisfiable, then G has a convex 2-cover. Note also that sets S_f and S_t are nontrivial and disjoint.

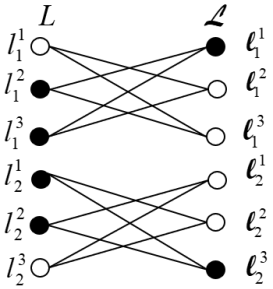


Figure 3.2. *Edges between L and $\mathscr{L}$.*

In Figure 3.1, we exhibit the graph G, which corresponds to a particular instance $(V, \mathscr{C}) = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\})$. Sets $Q \cup \mathscr{V} \cup \{f\}$ and $Q \cup \mathscr{V} \cup \{t\}$ are cliques in G. White vertices belong to S_t and black vertices belong to S_f. White vertices of $\mathscr{V}$ represent the variables of V set to true. All edges between L and $\mathscr{L}$ are represented in Figure 3.2. □

Now, taking into account the fact that the convex p-cover problem is NP-complete for a fixed $p \geq 3$, we can affirm that the convex p-cover problem is NP-complete for a fixed $p \geq 2$.

Corollary 3.1. *Convex 2-partition problem, nontrivial convex 2-cover problem, and nontrivial convex 2-partition problem are NP-complete.*

Proof. By construction of the particular graph G in the proof of the previous theorem, we conclude that G can be covered only by nontrivial convex sets, which are disjoint. So, every convex 2-cover is a convex 2-partition in G, and every convex 2-cover is a nontrivial convex 2-cover in G. □

3.2 Fixed number of nontrivial convex sets

In this section, we prove that the nontrivial convex p-cover problem is NP-complete for a fixed $p \geq 2$. We reduce the NP-complete clique p-partition problem [17] to the nontrivial convex p-partition problem.

Problem: Clique p-partition.
Instance: Graph $G = (X, U)$ and $p \in N$, $p \geq 3$.
Question: Is there a partition of X into p disjoint cliques?

Theorem 3.5. *The nontrivial convex p-partition problem is NP-complete for a fixed $p \geq 3$.*

Proof. By Theorem 3.3, verifying if a set is convex can be done in polynomial time. Thus, the nontrivial p-partition problem is in NP.

Let $G = (X, U)$ be a generic graph of clique p-partition problem. Without loss of generality, it can be assumed that X is not a clique. We generate a particular graph $G' = (X', U')$ of nontrivial convex p-partition problem from graph G by adding two auxiliary sets $Y = \{y_1, y_2, \ldots, y_p\}$ and $Z = \{z_1, z_2, \ldots, z_p\}$ to X such that $X' = X \cup Y \cup Z$, where $\Gamma(y_i) = X \cup \{z_i\}$ and $\Gamma(z_i) = X \cup \{y_i\}$ for each i, $1 \leq i \leq p$. Obviously, this construction of G' can be done in polynomial time. In Figure 3.3, it is shown how a particular graph G' of nontrivial convex p-partition problem is obtained from a graph G of clique p-partition problem.

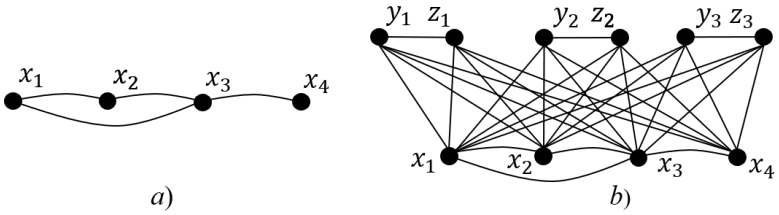


Figure 3.3. *The graph G' (case b)) that has a nontrivial convex 3-partition is obtained from the graph G (case a) that has a clique 3-partition.*

Graph G' satisfies the conditions of Theorem 2.3. Thus, every convex set of G' is a clique.

If $\mathcal{P}_p(G)$ is a clique p-partition of G, $p \geq 3$, then we can simply obtain a nontrivial convex p-partition $\mathcal{P}_p(G')$ of G' by addition the set $\{y_i, z_i\}$ to X_i for each i, $1 \leq i \leq p$, where $X_i \in \mathcal{P}_p(G)$.

On the other hand, if there is a nontrivial convex p-partition $\mathcal{P}_p(G')$, then we obtain a clique p-partition $\mathcal{P}_p(G)$ of G by subtracting the set $Y \cup Z$ from X'_i for each i, $1 \leq i \leq p$, where $X'_i \in \mathcal{P}_p(G')$. □

Corollary 3.2. *The nontrivial convex p-cover problem is NP-complete for a fixed $p \geq 3$.*

Proof. Again, it results from Theorem 3.3 that the convex p-cover problem is in NP.

Note that any proper convex set of the graph G' constructed in the previous theorem is a clique. Let $\mathcal{P}_p(G')$ be a nontrivial convex p-cover of graph G'. We obtain a family of sets $\mathcal{P} = \{X_1, X_2, \dots, X_p\}$ such that $X_i = X'_i \backslash (Y \cup Z)$, where $X'_i \in \mathcal{P}_p(G')$ for each i, $1 \leq i \leq p$. By removing from $\mathcal{P}$ all sets contained in the union of other sets of the family $\mathcal{P}$, we obtain a convex k-cover $\mathcal{P}_k(G)$ of G, where G is a graph of a clique p-partition problem and $k \leq p$. We can easily obtain a convex k-partition $\mathcal{P}'_k(G)$ by eliminating repeated vertices from sets of $\mathcal{P}_k(G)$. Note also that if any graph has a clique q-partition and a set S of this partition is not a singleton, then after the division of S into two cliques, we get a clique $(q+1)$-partition. Thus, G has a clique p-partition.

As we know from the previous theorem, every clique p-partition of a graph G, $p \geq 3$, implies the existence of nontrivial convex p-partition of G'. Now, if we recall that every nontrivial convex p-partition is a nontrivial convex p-cover, we deduce that every clique p-partition of G, $p \geq 3$, implies the existence of a nontrivial convex p-cover of G'.

So, the theorem is proved. □

Finally, the following two theorems are immediate consequences of Theorems 3.4, 3.5, and Corollaries 3.1 and 3.2.

Theorem 3.6. *The nontrivial convex p-cover problem is NP-complete for a fixed $p \geq 2$.*

Theorem 3.7. *The nontrivial convex p-partition problem is NP-complete for a fixed* $p \geq 2$.

3.3 Arbitrary number of nontrivial convex sets

The complexity of deciding whether a graph can be covered or partitioned by an arbitrary, but not less than two, number of nontrivial and convex sets is studied in the sequel. The first problem is formulated in the following way:

Problem: Nontrivial convex cover (NCC).

Instance: A graph $G = (X, U)$.

Question: Is there a nontrivial convex p-cover of G for some p, $2 \leq p \leq |X| - 2$?

Equivalently, it can be defined the nontrivial convex partition problem (NCP). For this purpose, we change only the question of NCC problem to this one: is there a nontrivial convex p-partition of G for some p, $2 \leq p \leq \left\lfloor \frac{|X|}{3} \right\rfloor$?

Note that if for every vertex of G, there exists at least one nontrivial convex set that contains it, then there is a nontrivial convex cover of G. The converse affirmation is also true. Based on these statements, we propose an algorithm represented below that solves the NCC problem in polynomial time.

Algorithm 3.1. *Determines whether a graph* $G = (X, U)$ *has a nontrivial convex p-cover for some* p, $2 \leq p \leq |X|$.

Input: *A graph* $G = (X, U)$.

Output: *Nontrivial convex cover* $\mathcal{P}(G)$ *or nothing.*

1: $\mathcal{P}(G) \leftarrow \emptyset$
2: $M \leftarrow \emptyset$
3: **for** $x \in X$ **do**
4: **if** $x \notin M$ **then**
5: $flag \leftarrow 0$
6: **for** $y, z \in X \backslash \{x\}$, $y \neq z$ **do**
7: $S \leftarrow d - conv(\{x, y, z\})$
8: **if** $S \neq X$ **then**

9: $\mathcal{P}(G) \leftarrow \mathcal{P}(G) \cup \{S\}$
10: $M \leftarrow M \cup S$
11: $flag \leftarrow 1$
12: **break**
13: **if** $flag = 0$ **then**
14: **stop**: there does not exist any nontrivial convex set containing x
15: **for** $S \in \mathcal{P}(G)$ **do**
16: **if** $S \subseteq \bigcup_{C \in \mathcal{P}(G), C \neq S} C$ **then**
17: $\mathcal{P}(G) \leftarrow \mathcal{P}(G) \backslash \{S\}$
18: **return** $\mathcal{P}(G)$

Theorem 3.8. *Algorithm 3.1 decides in $O(n^4 m)$ time whether a graph $G = (X, U)$ can be covered by nontrivial convex sets.*

Proof. Steps 1, 2 and 18 run in constant time. Steps 3 – 14 determine, for every vertex $x \in X$, if there exists a nontrivial convex set $S \subset X$, $x \in S$. If there is such a set S, then there are at least two different vertices $y, z \in X$, $y \neq x$, $z \neq x$, for which $d - conv(\{x, y, z\}) \subseteq S$. Consequently, it is sufficient to build the convex hull for all sets consisting of three vertices, one of which is x. According to Theorem 3.3, a convex hull of a set $S \subset X$ can be constructed in time $O(|d - conv(S)|m)$. Since $d - conv(S)$ potentially can contain all n vertices, steps 3 – 14 run in $O(n^4 m)$ time.

Steps 15 – 17 exclude from $\mathcal{P}(G)$ all sets contained in the union of other sets of $\mathcal{P}(G)$. So, we obtain a nontrivial convex cover of G. The resulting family $\mathcal{P}(G)$ has at most $n - 2$ sets, and every set of $\mathcal{P}(G)$ contains no more than $n - 1$ vertices. Further, steps 15 – 17 run in $O(n^3)$ time and the complexity of the whole algorithm is $O(n^4 m)$. □

We prove that the NCP problem is NP-complete by reduction from the well-known NP-complete partition into triangles problem [18], formulated as follows:

Problem: Partition into triangles (PIT).

Instance: A graph $G = (X, U)$ with $|X| = 3q$, where $q \in N$.

Question: Is there a partition of X into q disjoint subsets $X_1, X_2, \ldots, X_q$ of size 3 such that each X_i, $1 \leq i \leq q$, induces a triangle in graph G?

Note that the PIT problem remains NP-complete even if the input graph G is tripartite [19]. Moreover, it is evident that every tripartite graph has no cliques of cardinality 4.

Theorem 3.9. *The NCP problem is NP-complete.*

Proof. Taking into account Theorem 3.3, we see that the NCP problem is in NP.

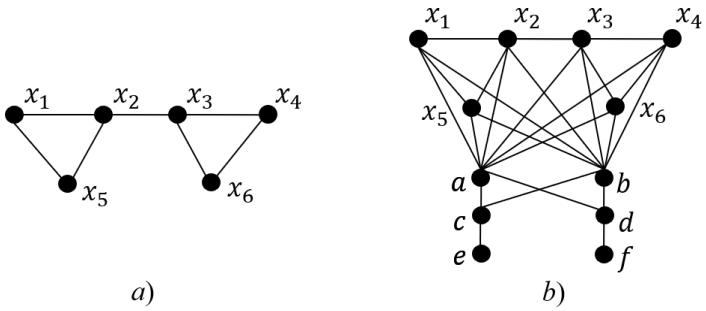
We reduce the NP-complete PIT problem for a tripartite graph to the NCP problem. Let $G = (X, U)$ be a tripartite graph of PIT problem, $|X| = 3q$, $q \in N$. We construct a graph $G' = (X', U')$ of NCP problem from G in the following way:

1) $X' = X \cup \{a, b, c, d, e, f\}$;

2) $U' = U \cup \{(a, c), (a, d), (b, c), (b, d), (c, e), (d, f)\} \cup$

$\cup \{(a, x), (b, x) : x \in X\}$.

Evidently, this construction of G' can be done in polynomial time. In Figure 3.4, the graph G' (case b) that corresponds to an instance of PIT problem G (case a) is represented.



a) *b*)

Figure 3.4. *The graph G' (case b)) that admits a nontrivial convex partition is obtained from the graph G (case a) that admits a partition into triangles.*

We have to show that G admits a partition into q triangles if and only if there exists a nontrivial convex partition of G'.

Let $\mathcal{P}_q(G) = \{X_1, X_2, \ldots, X_q\}$ be a family of triangles that partitions G. Since every triangle is a clique in G, it follows that each set X_i, $1 \leq i \leq q$, is nontrivial convex in G', and the set of vertices $\{a, b, c, d, e, f\}$ remains uncovered in G'. We observe that $d - conv_{G'}(\{a, c, e\}) = \{a, c, e\}$ and

$d - conv_{G'}(\{b,d,f\}) = \{b,d,f\}$. So, $\mathcal{P}_q(G) \cup \{\{a,c,e\},\{b,d,f\}\}$ formes a partition of G' into $q+2$ nontrivial convex sets.

Let $\mathcal{P}(G')$ be a partition of G' into nontrivial convex sets and let S be a set of $\mathcal{P}(G')$. We distinguish some properties of S:

1) $\{a,b\} \not\subset S$. Assuming the contrary, namely that $\{a,b\} \subset S$, we get $d - conv_{G'}(\{a,b\}) = X \cup \{a,b,c,d\} \quad X' \backslash d - conv_{G'}(\{a,b\}) = \{e,f\}$. Note that the set $\{e,f\}$ is not nontrivial. Hence, $\mathcal{P}(G')$ can not partition G' into nontrivial convex sets. We get the required contradiction.

2) $\{c,d\} \not\subset S$. Assuming the converse, $\{a,b\} \subset d - conv_{G'}(\{c,d\})$. Therefore, property 1) is not satisfied, and we obtain a contradiction.

3) $\{e,f\} \not\subset S$. In the converse case, $\{a,b,c,d\} \subset d - conv_{G'}(\{e,f\})$ and, consequently, properties 1) and 2) are not satisfied. This implies a contradiction.

4) $\{x,y\} \not\subset S$ for every vertex $x \in X$ and $y \in \{c,d\}$. Assuming the converse, namely that $\{x,y\} \subset S$, we get a contradiction because $\{a,b\} \subset d - conv_{G'}(\{x,y\})$.

5) $\{x,y\} \not\subset S$ for every two nonadjacent vertices $x,y \in X$. In the converse case, if $\{x,y\} \subset S$, then $\{a,b\} \subset d - conv_{G'}(\{x,y\})$. Have a contradiction.

Let $S_1 = \{a,c,e\}$, $S_2 = \{b,d,f\}$, $S_3 = \{b,c,e\}$ and $S_4 = \{a,d,f\}$. Taking into account properties 1) – 5) and the fact that every vertex of X' belongs exactly to one set of $\mathcal{P}(G')$, it is seen that the family $\mathcal{P}(G')$ contains exactly a pair of sets of the following two: S_1, S_2 or S_3, S_4. Each pair of these sets covers vertices a,b,c,d,e,f. Hence, vertices of the set $X' \backslash \{a,b,c,d,e,f\}$ remain to be partitioned into nontrivial convex sets. By property 5), all of these sets have to be cliques. As mentioned above, G has no cliques with $k \geq 4$ vertices. Further, all of these sets induce triangles, and by the elimination of a pair of sets S_1, S_2 or S_3, S_4 from $\mathcal{P}(G')$, we obtain a family of triangles $\mathcal{P}_q(G)$. □

3.4 Invariants

We show that it is NP-hard to determine the values of the invariants $\varphi_c^{min}(G)$, $\theta_c^{min}(G)$, $\varphi_{cn}^{min}(G)$, $\theta_{cn}^{min}(G)$, $\varphi_{cn}^{max}(G)$, and $\theta_{cn}^{max}(G)$ for a graph G. For this purpose, we prove that the related decision problems are NP-complete.

Determination of the minimum convex cover number $\varphi_c^{min}(G)$ has the following decision problem:

Problem: Minimum convex cover (MinCC).
Instance: Graph $G = (X, U)$, integer p, $2 \leq p \leq |X|$.
Question: Is there a convex q-cover of G such that $2 \leq q \leq p$?

As for invariants $\theta_c^{min}(G)$, $\varphi_{cn}^{min}(G)$, $\theta_{cn}^{min}(G)$, their decision problems can be defined in the same manner, and only the appropriate specification of the type of convex cover (convex q-partition, nontrivial convex q-cover, nontrivial convex q-partition) in the questions is required. We will refer to these problems as: minimum convex partition (MinCP), minimum nontrivial convex cover (MinNCC) and minimum nontrivial convex partition (MinNCP).

Theorem 3.10. *The MinCC problem is NP-complete.*

Proof. From Theorem 3.3, we know that verifying whether a set of vertices is convex can be done in polynomial time. Hence, the MinCC problem is in NP.

Let $G = (X, U)$ be a generic graph of clique p-partition problem (see section 3.2) and p be an integer, $3 \leq p \leq |X|$. Without loss of generality, it can be assumed that X is not a clique. We construct a particular graph $G' = (X', U')$ of MinCC problem from graph G similarly as in the proof of Theorem 3.5. Obviously, this construction of G' can be done in polynomial time.

It can be checked that for every two nonadjacent vertices $a, b \in X'$, we get $X' \subseteq d - conv(\{a, b\})$. Consequently, a set $S \subset X'$ is convex in G' if and only if S is a clique, and further G' cannot be covered by k, $k < p$, convex sets.

As in the proof of Theorem 3.5 and Corollary 3.2, it can be easily shown that G can be partitioned into p disjoint cliques if and only if there exists a cover of G' by at most p convex sets, where $p \geq 3$ is fixed. At the same time, by Theorem 3.4, the MinCC problem is NP-complete for $p = 2$. Combining these two facts, we conclude that the MinCC problem is NP-complete. □

In view of the demonstration of Theorem 3.10, we obtain the correctness of Corollaries 3.3 – 3.5.

Corollary 3.3 *The MinCP problem is NP-complete.*

Corollary 3.4 *The MinNCC problem is NP-complete.*

Corollary 3.5 *The MinNCP problem is NP-complete.*

Determination of the maximum nontrivial convex partition number $\theta_{cn}^{max}(G)$ has the following decision problem:

Problem: Maximum nontrivial convex partition (MaxNCP).

Instance: Graph $G = (X, U)$, integer p, $2 \leq p \leq |X|$

Question: Is there a nontrivial convex q-partition of G such that $q \geq p$?

Theorem 3.11. *The MaxNCP problem is NP-complete.*

Proof. Notice that the MaxNCP problem is in NP because, by Theorem 3.3, verifying whether a set of vertices is convex can be done in polynomial time.

Let $G = (X, U)$ be an instance of PIT problem (see section 3.3), $|X| = 3k$, $k \in N$. We construct a particular graph $G' = (X', U')$ of MaxNCP problem that corresponds to G similarly as in the proof of Theorem 3.9. It is easy to see that G' can be constructed in polynomial time.

The MaxNCP instance is defined by the graph G' and a number $p = k + 2$. Likewise in Theorem 3.9, it can be easily shown that there exists a partition of G into triangles if and only if there exists a nontrivial convex partition of G' of size at least $k + 2$. □

The decision problem for maximum nontrivial convex cover number $\varphi_{cn}^{max}(G)$ is formulated similarly to MaxNCP, but with a different question: Is there a nontrivial convex q-cover of G such that $q \geq p$? We will refer to this problem as maximum nontrivial convex cover (MaxNCC).

We establish the NP-completeness of MaxNCC problem by reduction from the 3-Satisfiability problem that is known to be NP-complete [18].

The 3-Satisfiability problem is defined as follows:

Problem: 3-Satisfiability (3SAT).

Instance: Given a boolean expression E in conjunctive normal form that is the conjunction of clauses, each of which is the disjunction of three distinct literals.

Question: Is there a satisfying truth assignment for E?

Theorem 3.12. *The MaxNCC problem is NP-complete.*

Proof. It is known from Theorem 3.3 that verifying whether a set of vertices is convex can be done in polynomial time. Thus, the MaxNCC problem is in NP.

The following reduction from 3SAT to MaxNCC will establish that MaxNCC problem is NP-complete. Let E be an instance of the 3SAT problem with n variables $V_1, V_2, \dots, V_n$ and m clauses $K_1, K_2, \dots, K_m$. Given this instance, we construct a graph $G = (X, U)$ with $3n + m + 16$ vertices. Vertices v_i, $\overline{v_i}$, y_i correspond to variable V_i, $1 \leq i \leq n$. One vertex k_j corresponds to clause K_j, $1 \leq j \leq m$. There are supplementary vertices grouped in the four sets: $A = \{a, a_1, a_2, a_3\}$, $B = \{b, b_1, b_2, b_3\}$, $C = \{c, c_1, c_2, c_3\}$ and $\{d, e, f, h\}$. We denote by V, $\overline{V}$ and Y sets of all vertices v_i, $\overline{v_i}$ and respectively y_i, $1 \leq i \leq n$. By K we denote the set of all vertices k_j, $1 \leq j \leq m$.

The graph G has $12n + 6m + 27$ edges. The three vertices corresponding to each variable are connected by an edge (i.e., v_i, $\overline{v_i}$, y_i form a triangle). Each clause vertex is connected to its component terms (that is, the clause vertex k_l corresponding to the clause $\overline{V_i} \vee V_j \vee V_q$ is connected by edges to vertices $\overline{v_i}, v_j, v_q$. The three additional vertices a, b, c are connected to v_i, $\overline{v_i}$, y_i for all i, $1 \leq i \leq n$, and to k_j for all j, $1 \leq j \leq m$. The vertex d is connected by an edge to each vertex $r \in (A \cup B \cup \{h\}) \backslash \{a_1, b_1\}$, the vertex e is connected by an edge to each vertex $r \in (B \cup C \cup \{h\}) \backslash \{b_1, c_1\}$ and f is connected to each vertex $r \in (A \cup C \cup \{h\}) \backslash \{a_1, c_1\}$. Finally, a_1 is connected to a_2 and a_3, b_1 is connected to b_2 and b_3, c_1 is connected to c_2 and c_3.

The MaxNCC instance is defined by the graph G and a number $p = 2n + m + 3$. It should be clear that this construction of G can be done in polynomial time.

For example, we consider the 3SAT instance:

$$E = (V_1 \vee V_2 \vee \overline{V_3}) \wedge (V_2 \vee V_3 \vee \overline{V_4}) \wedge (V_1 \vee \overline{V_2} \vee V_4)$$

Then the graph G corresponding to E is presented in Figure 3.5.

Without loss of generality, we consider that E has no clauses which contain a variable and its negation.

Let us distinguish some properties of $G = (X, U)$:

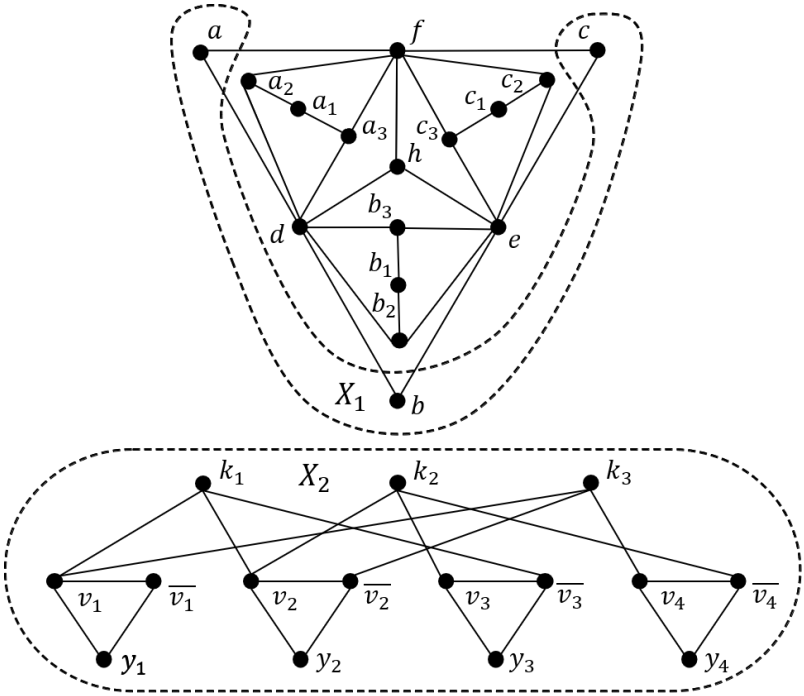


Figure 3.5. *Graph G corresponds to the instance $E = (V_1 \vee V_2 \vee \overline{V_3}) \wedge (V_2 \vee V_3 \vee \overline{V_4}) \wedge (V_1 \vee \overline{V_2} \vee V_4)$. Every vertex of set X_1 is adjacent to every vertex of set X_2. Dashed line marks areas of sets X_1 and X_2.*

Property 1: For any two nonadjacent vertices $x, y \in V \cup \overline{V} \cup Y \cup K \cup \{a, b, c\}$, we have $d - conv(\{x, y\}) = X$.

Property 2: For any two vertices $x \in V \cup \overline{V} \cup Y \cup K$, $y \in (A \cup B \cup C \cup \{d, e, f, h\}) \backslash \{a, b, c\}$, we have $d - conv(\{x, y\}) = X$.

Property 3: Let $\mathcal{P}(G)$ be a nontrivial convex cover or G. Then, $A^* = A \cup \{d, f, h\}$, $B^* = B \cup \{d, e, h\}$ and $C^* = C \cup \{e, f, h\}$ are in $\mathcal{P}(G)$.

The correctness of the first two properties can be easily verified. We will show that Property 3 is correct too. Let $S \subset X$ be a nontrivial convex set that contains the vertex a_1. Since $|S| \geq 3$, the structure of graph G yields that S also contains vertices a_2 and a_3. Hence, we get the equality $d - conv(\{a_2, a_3\}) = A^*$. Note that for each vertex $x \in X \backslash A^*$, we have $d - conv(A^* \cup \{x\}) = X$. Further, A^* is a unique nontrivial convex set different from X that contains a_1. For the same reason, B^* and C^* are

unique nontrivial convex sets different from X, which contain vertices b_1 and c_1, respectively.

We have to show that E is satisfiable if and only if there exists a nontrivial convex cover of G of size at least $2n + m + 3$.

Assume E is satisfiable. Then there exists a truth assignment of variables such that all clauses evaluate to true. We will form a nontrivial convex cover of $\mathcal{P}(G)$ as follows. By Property 3, $\mathcal{P}(G)$ includes the sets A^*, B^* and C^*. We denote by M the set of all vertices v_i for which V_i are true in the assignment, and all $\overline{v_i}$ for which V_i are false. $\mathcal{P}(G)$ includes sets $\{x, y, a\}$ for all $x \in M$, where $y \in Y$ and $y \sim x$. Also, $\mathcal{P}(G)$ includes sets $\{v_i, \overline{v_i}, a\}$ for each i, $1 \leq i \leq n$. Moreover, $\mathcal{P}(G)$ includes one set $\{x, k_j, a\}$ for each k_j, where $x \in M$, $x \sim k_j$, and the existence of such a vertex x yields from the fact that E is satisfiable. The obtained nontrivial convex cover $\mathcal{P}(G)$ contains exactly $2n + m + 3$ sets.

Assume there exists a nontrivial convex cover $\mathcal{P}(G)$ of size at least $2n + m + 3$. Taking into account the Property 3, $\mathcal{P}(G)$ already includes the sets A^*, B^* and C^*. In view of Property 2, it remains to analyze a nontrivial convex cover $\mathcal{P}(G')$ resulted from $\mathcal{P}(G)$ after the elimination of sets A^*, B^* and C^*, where $G' = G[V \cup \overline{V} \cup Y \cup K \cup \{a, b, c\}]$. Clearly, $|\mathcal{P}(G)| = |\mathcal{P}(G')| + 3$. It follows from the Property 1 and the structure of G that every set $S \in \mathcal{P}(G')$ is a clique, $3 \leq |S| \leq 4$, and it can be classified into one of three types:

(i) $S = \{x, y, z\}$, where $x, y \in V \cup \overline{V} \cup Y \cup K$, $x \sim y$, and $z \in \{a, b, c\}$;

(ii) $S = \{v_i, \overline{v_i}, y_i\}$ for any i, $1 \leq i \leq n$;

(iii) $S = \{v_i, \overline{v_i}, y_i, x\}$ for any i, $1 \leq i \leq n$, where $x \in \{a, b, c\}$.

Now we define a family of convex sets $\mathcal{P}(G'') = \emptyset$, obtained from $\mathcal{P}(G')$, that will cover $G'' = G'[V \cup \overline{V} \cup Y \cup K]$.

We examine each set $S \in \mathcal{P}(G')$ and consider two cases.

1) If S is of the first type (i), then we add set $S \backslash \{a, b, c\}$ to $\mathcal{P}(G'')$.

2) If S is of the second (ii) or the third (iii) type, then taking into account the fact that vertices a, b, and c are already covered by sets A^*, B^* and C^*, we analyze two options. Suppose S contains only one resident vertex r in $\mathcal{P}(G')$. In this case, we add set $\{r, x\}$ to $\mathcal{P}(G'')$, where vertex

$x \in S\backslash\{a, b, c\}$, $x \neq r$. Suppose S contains at least two resident vertices r_1, r_2 in $\mathcal{P}(G')$. Then, we add sets $\{r_1, x\}$ and $\{r_2, x\}$ to $\mathcal{P}(G'')$, where vertex $x \in S\backslash\{a, b, c\}$, $x \neq r_1$, $x \neq r_2$.

Let us remark that for a set $S \in \mathcal{P}(G')$ of the type (ii) or (iii), any set $S' \in \mathcal{P}(G')$, $S' \neq S$, $S' \cap S\backslash\{a, b, c\} \neq \emptyset$, is of the first type (i), and thus there are no uncovered vertices in G'', i.e., $\mathcal{P}(G'')$ is a convex cover of G''. Clearly, $|\mathcal{P}(G'')| \geq |\mathcal{P}(G')|$ and thus $|\mathcal{P}(G'')| \geq |\mathcal{P}(G)| - 3 = 2n + m$. We need to take a closer look at the family $\mathcal{P}(G'')$. Every set of $\mathcal{P}(G'')$ has exactly two adjacent vertices. We choose one resident vertex of each set $S \in \mathcal{P}(G'')$ and form the set W as a union of these vertices. It is clear that $D = X(G'')\backslash W$ is a dominating set of G'' such that $|\mathcal{P}(G'')| + |D| = |V \cup \overline{V} \cup Y \cup K| = 3n + m$. If we combine this with the previous inequality, we get $|D| \leq n$.

Consider the graph G''. For any i, $1 \leq i \leq n$, y_i is either in D or adjacent to a vertex of D, and y_i is connected by edges only to v_i and $\overline{v_i}$. It follows that for every i, $1 \leq i \leq n$, either v_i, $\overline{v_i}$, or y_i is in D. This already specifies n vertices, so exactly one vertex for each variable is included in D. We create a truth assignment as follows. V_i will be assigned to true if v_i is in D. Otherwise, V_i will be assigned to false. Consider clause K_j. The vertex k_j is not in the dominating set. So k_j is adjacent to some v_i or $\overline{v_l}$ in the dominating set. If k_j is adjacent to v_i in D, then since V_i is set to true, it follows that k_j will be true. If k_j is adjacent to $\overline{v_l}$ in D, then v_l is not in D and V_l will be false, so K_j will be true. It follows that this assignment is a solution for E and E is satisfiable. Thus if G has a nontrivial convex cover $\mathcal{P}(G)$ of size at least $2n + m + 3$ then E is satisfiable.

So, this completes the proof of the correctness of the reduction, and we conclude that MaxNCC is NP-complete. □

Chapter 4
Convex 2-Covers

We define two special types of convex 2-covers of graphs. A family of two sets $\mathcal{P}_{2,t}(G) = \{S_t, S_{nt}\}$ is said to be a *convex* $(2,t)$*-cover* of a graph G if it is a convex 2-cover of G and S_t is a trivial set. In the same way, $\mathcal{P}_{2,nt}(G) = \{S_1, S_2\}$ is said to be a *convex* $(2,nt)$*-cover* of a graph G if it is a convex 2-cover of G such that S_1 and S_2 are both nontrivial convex sets. A family of all possible convex $(2,t)$-covers of G is denoted by $\widetilde{\mathcal{P}}_{2,t}(G) = \{P^1_{2,t}(G), P^2_{2,t}(G), \dots, P^k_{2,t}(G)\}$, $k \geq 0$.

Theorem 3.4 says that it is NP-complete to decide whether a graph G has a convex $(2,nt)$-cover. By Theorem 3.3, we know that verifying whether a set is convex can be done in polynomial time. Consequently, determining if there exists a convex $(2,t)$-cover of a graph also can be done in polynomial time.

A series of results related to establishing the existence of a convex $(2,nt)$-cover of a graph in dependency on existing convex $(2,t)$-covers are presented in this chapter.

4.1 Convex $(2,t)$-covers and convex $(2,nt)$-covers

It is absolutely clear that every connected graph G on $n = 2$ or $n = 3$ vertices has a convex $(2,t)$-cover, but has no convex $(2,nt)$-cover. If $n = 4$, then as a consequence of Theorem 2.7, we get the following corollary.

Corollary 4.1. *If G is a connected graph of order* 4*, then G has a convex $(2,nt)$-cover if and only if $G \neq C_4$.*

Theorem 4.1. *Let $G = (X, U)$, $|X| \geq 5$, be a connected graph. Then the following conditions are equivalent:*

a) there exists a simplicial vertex $x \in X$;

b) there exists a family $\mathcal{P}_{2,t}(G) = \{S_t = \{x\}, S_{nt} = X\backslash\{x\}\}$;

c) there exists a family $\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X\backslash\{x\}\}$.

Proof. Since x is a simplicial vertex in G, every two vertices $y, z \in \Gamma(x)$ are adjacent, $d - conv(\Gamma(x)) = \Gamma(x)$ and $d - conv(X\backslash\{x\}) = X\backslash\{x\}$. Thus, G can be covered by a convex $(2, t)$-cover

$$\mathcal{P}_{2,t}(G) = \{S_t = \{x\}, S_{nt} = X\backslash\{x\}\}.$$

Consequently, 1) ⇒ 2).

Suppose there exists $\mathcal{P}_{2,t}(G) = \{S_t = \{x\}, S_{nt} = X\backslash\{x\}\}$. Since G is connected, there is at least one vertex y such that $y \sim x$ and $d - conv(\{x, y\}) = \{x, y\}$. Therefore, G can be covered by

$$\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X\backslash\{x\}\}.$$

Consequently, 2) ⇒ 3).

Suppose there exists $\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X\backslash\{x\}\}$. Since S_{nt} is convex, $\Gamma(x)$ is a clique in G. Whence, x is a simplicial vertex. Consequently, 3) ⇒ 1). □

Theorem 4.2. *Let $G = (X, U)$, $|X| \geq 5$, be a connected graph that contains a simplicial vertex. Then G has a convex $(2, nt)$-cover.*

Proof. It follows from Theorem 4.1 that there is a convex $(2, t)$-cover $\mathcal{P}_{2,t}(G) = \{S_t = \{x\}, S_{nt} = X\backslash\{x\}\}$, where x is a simplicial vertex. We consider 2 cases.

1) $\Gamma(x) = \{y\}$. Since G is a connected graph and $|X| - 1 > 3$, there exists a vertex $z \in S_{nt}$ such that $z \sim y$. Then according to $\langle x, z\rangle = \{x, y, z\}$ and $d - conv(\{x, y, z\}) = \{x, y, z\}$, G has a convex $(2, nt)$-cover

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{x, y, z\}, S_2 = S_{nt}\}.$$

2) $|\Gamma(x)| \geq 2$. We select two different vertices $y, z \in \Gamma(x)$. Since x is a simplicial vertex, $y \sim z$ and $\{x, y, z\}$ is a triangle that is a nontrivial convex set. This implies that G has a convex $(2, nt)$-cover

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{x, y, z\}, S_2 = S_{nt}\}.$$

Finally, G has a convex $(2, nt)$-cover. □

Theorem 4.3. *Let $G = (X, U)$, $|X| \geq 5$, be a connected graph without simplicial vertices. Then the following conditions are equivalent:*

1) there exist two adjacent vertices $x, y \in X$ such that $A = \Gamma(x)\backslash\{y\}$ and $B = \Gamma(y)\backslash\{x\}$ are cliques in G, where for any two vertices $a \in A$ and $b \in B$ the inequality $d(a, b) \leq 2$ is satisfied;

2) there exists a family $\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X\backslash\{x, y\}\}$.

Proof. Combining Theorem 4.1 with the absence of simplicial vertices in G, we get that G has no convex $(2, t)$-cover such that the cardinality of the trivial set is one, or the cardinality of the trivial set is two and it has a common vertex with the nontrivial set.

Let $x, y \in X$ be two vertices, which satisfy condition 1). Then the following relations hold:

$$d - conv(\{x, y\}) = \{x, y\},$$
$$\{x, y\} \cap d - conv(A \cap B) = \emptyset.$$

It follows that G has a convex $(2, t)$-cover

$$\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X\backslash\{x, y\}\}.$$

Consequently, 1) ⇒ 2).

Suppose now that there exists a convex $(2, t)$-cover $\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X\backslash\{x, y\}\}$. According to the theorem conditions, G does not contain simplicial vertices. Since sets S_t and S_{nt} are convex, $x \sim y$ and sets $A = \Gamma(x)\backslash\{y\}$, $B = \Gamma(y)\backslash\{x\}$ are cliques in G. Moreover, if there exist two vertices $a \in A$ and $b \in B$ such that $d(a, b) > 2$, then $\{x, y\} \subseteq \langle a, b\rangle \subseteq S_{nt}$. Have a contradiction. Further, this means that for any two vertices $a \in A$ and $b \in B$, we get $d(a, b) \leq 2$. It follows that 2) ⇒ 1). □

Theorem 4.4. *Let $G = (X, U)$, $|X| \geq 5$, be a connected graph without simplicial vertices and let $\widetilde{\mathcal{P}}_{2,t}(G)$ contains two convex $(2, t)$-covers such that the intersection of their trivial sets is empty. Then G has a convex $(2, nt)$-cover.*

Theorem 4.4 follows from the fact that the nontrivial convex sets from the respective convex $(2, t)$-covers form a convex $(2, nt)$-cover of G.

Theorem 4.5. *Let $G = (X, U)$, $|X| \geq 5$, be a connected graph without simplicial vertices and let $|\widetilde{\mathcal{P}}_{2,t}(G)| = k \geq 2$ such that the intersection of the trivial sets S_t^i of any two convex $(2, t)$-covers is nonempty. Then exactly one of the following conditions holds:*

1) $|\widetilde{\mathcal{P}}_{2,t}(G)| = 3$ and $S_t^1 \cup S_t^2 \cup S_t^3$ forms a triangle in G;

2) $|\bigcap_{i=1}^{k} S_t^i| = 1$.

Proof. Since G has no simplicial vertices, using Theorem 4.1, we get that the cardinality of the trivial set of each convex $(2, t)$-cover of G is two, and every trivial set does not intersect the corresponding nontrivial convex set. Let us consider 3 cases.

$|\widetilde{\mathcal{P}}_{2,t}(G)| = 2$. It follows that $|S_t^1 \cap S_t^2| = 1$. Hence, condition 2) is satisfied.

$|\widetilde{\mathcal{P}}_{2,t}(G)| = 3$. If $|S_t^1 \cap S_t^2 \cap S_t^3| = 1$, then condition 2) is satisfied. Otherwise, $S_t^1 \cup S_t^2 \cup S_t^3$ forms a triangle in G and, as a result, condition 1) is satisfied.

$|\widetilde{\mathcal{P}}_{2,t}(G)| = k \geq 4$. Obviously, in this case, we have $|\bigcap_{i=1}^{k} S_t^i| = 1$. This means that condition 2) is satisfied. □

Theorem 4.6. *Let $G = (X, U)$, $|X| \geq 5$, be a connected graph without simplicial vertices that satisfies the equality:*

$$\widetilde{\mathcal{P}}_{2,t}(G) = \left\{P_{2,t}^i(G) = \left\{S_t^i, S_{nt}^i\right\} : 1 \leq i \leq 3\right\},$$

where $S_t^1 \cup S_t^2 \cup S_t^3$ forms a triangle in G. Then G has a convex $(2, nt)$-cover.

Proof. Let $S = S_t^1 \cup S_t^2 \cup S_t^3$. It is obvious that G can be covered by one of the three convex $(2, nt)$-covers:

$$\mathcal{P}_{2,nt}^1(G) = \{S_{nt}^1, S\}, \mathcal{P}_{2,nt}^2(G) = \{S_{nt}^2, S\}, \mathcal{P}_{2,nt}^3(G) = \{S_{nt}^3, S\}.$$

This proves the theorem. □

Theorem 4.7. *Let $G = (X, U)$, $|X| \geq 5$, be a connected graph without simplicial vertices that satisfies the equality:*

$$\widetilde{\mathcal{P}}_{2,t}(G) = \left\{P_{2,t}^i(G) = \left\{S_t^i = \{a, b_i\}, S_{nt}^i\right\} : 1 \leq i \leq k, k \geq 3\right\}.$$

Then G has a convex $(2, nt)$-cover.

Proof. According to the theorem conditions, we have $|\widetilde{\mathcal{P}}_{2,t}(G)| \geq 3$, $\bigcap_{i=1}^{k} S_t^i = \{a\}$ and $|\Gamma(a)\backslash\{b_i\}| \geq 2$ for each i, $1 \leq i \leq k$. Combining the absence of simplicial vertices in G with Theorem 4.1, we get that the cardinality of the trivial set of each convex $(2,t)$-cover of G is two, and every trivial set does not intersect the corresponding nontrivial convex set. Now, $b_i \sim b_j$ for any two $i,j \in \{1,2,\dots,k\}$, $i \neq j$, because $a \notin S_{nt}^l$, $1 \leq l \leq k$. Hence, $\bigcup_{i=1}^{k} S_t^i$ is a nontrivial clique in G. Furthermore, set $\bigcup_{i=1}^{k} S_t^i$ is nontrivial and convex. There exists a convex $(2,nt)$-cover of graph G:

$$\mathcal{P}_{2,nt}(G) = \left\{\{a, b_1, b_2\}, S_{nt}^1\right\}.$$

This proves the theorem. □

4.2 Class $\mathcal{F}$

Now we define a class of graphs $\mathcal{F}$ consisting of graphs $G = (X, U)$ that satisfy the following conditions:

1) $X = \{a, b_1, b_2, x_1, x_2, \dots, x_m\}$, $m \geq 1$;

2) $U = \{(a, b_1), (a, b_2)\} \cup \left\{\left(x_i, x_j\right) : 1 \leq i, j \leq m; i \neq j\right\} \cup$
$\cup \{(b_1, x_i), (b_2, x_i) : 1 \leq i \leq m\}$.

It can easily be checked that any graph $G \in \mathcal{F}$ on $n \geq 5$ vertices has exactly two convex $(2,t)$-covers:

$$\mathcal{P}_{2,t}^1(G) = \left\{\{a, b_1\}, \{b_2, x_1, \dots, x_m\}\right\}, \mathcal{P}_{2,t}^2(G) = \left\{\{a, b_2\}, \{b_1, x_1, \dots, x_m\}\right\}.$$

The graph class $\mathcal{F}$ is presented in Figure 4.1.

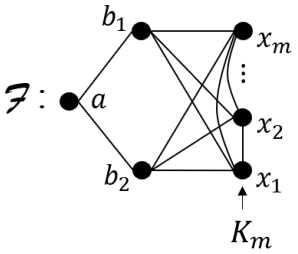


Figure 4.1. *The graph class $\mathcal{F}$.*

Theorem 4.8. *A graph $G = (X, U) \in \mathcal{F}$ has no convex $(2, nt)$-cover.*

Proof. By definition, $|X| \geq 4$. If $|X| = 4$, then $G = C_4$. Under the conditions of Corollary 4.1, G has no convex $(2, nt)$-cover.

Suppose $|X| \geq 5$. Assume that graph G has a convex $(2, nt)$-cover. Further, one of the nontrivial convex sets of this convex $(2, nt)$-cover contains vertices $\{b_1, b_2\}$ or $\{a, x\}$, where $x \in X \backslash \{a, b_1, b_2\}$. Notice that for every graph $G = (X, U) \in \mathcal{F}$, the following conditions hold:

$$\{b_1, b_2\} \subset \langle a, x \rangle \text{ for each } x \in X \backslash \{a, b_1, b_2\};$$

$$d - conv(\{b_1, b_2\}) = X.$$

This contradiction proves the theorem. □

Theorem 4.9. *Let $G = (X, U)$, $|X| \geq 5$, $G \notin \mathcal{F}$, be a connected graph without simplicial vertices that satisfies the equality:*

$$\widetilde{\mathcal{P}}_{2,t}(G) = \left\{P^1_{2,t}(G) = \{S^1_t = \{a, b_1\}, S^1_{nt}\}, P^2_{2,t}(G) = \{S^2_t = \{a, b_2\}, S^2_{nt}\}\right\}.$$

Then G has a convex $(2, nt)$-cover.

Proof. Suppose $b_1 \sim b_2$. Then G has convex $(2, nt)$-covers:

$$\mathcal{P}^1_{2,nt}(G) = \left\{\{a, b_1, b_2\}, S^1_{nt}\right\},$$

$$\mathcal{P}^2_{2,nt}(G) = \left\{\{a, b_1, b_2\}, S^2_{nt}\right\}.$$

Now suppose that b_1 is not adjacent to b_2. Denote $A = \Gamma(a) \backslash \{b_1\}$ and $B = \Gamma(b_1) \backslash \{a\}$. We see that $A, B \subseteq S^1_{nt}$. If $S^1_{nt} \neq d - conv(A \cup B)$, then G has a convex $(2, nt)$-cover

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{a, b_1\} \cup d - conv(A \cup B), S_2 = S^1_{nt}\}.$$

Assume that $S^1_{nt} = d - conv(A \cup B)$. In addition, suppose that $|A| \geq 2$. It follows from Theorem 4.3 that $A \cup \{a\}$ is a clique in G. Thus, $A \cup \{a\}$ is a nontrivial convex set. By the theorem conditions, $b_2 \in A$, $b_1 \in S^2_{nt}$. Hence, G has a convex $(2, nt)$-cover

$$\mathcal{P}_{2,nt}(G) = \{S_1 = A \cup \{a\}, S_2 = S^2_{nt}\}.$$

Further, assume that $A = \{b_2\}$. In accordance with Theorem 4.3, we obtain that $|B| \geq 1$. Let us consider 2 cases.

1) Suppose $S^1_{nt} \neq A \cup B$. Then, combining the fact that S^1_{nt} is convex with Theorem 4.3, there is a vertex $x \in B$ that satisfies $d(b_2, x) = 2$ such

that there is a vertex $y \in \langle b_2, x\rangle$, where $y \notin A \cup B$, $y \in d - conv(A \cup B)$; otherwise $S_{nt}^1 = A \cup B$. This implies that G has a convex $(2, nt)$-cover

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{a, b_1, x\}, S_2 = S_{nt}^1\}.$$

2) Suppose $S_{nt}^1 = A \cup B$. Then, since $|X| \geq 5$ and $|A| = 1$, it follows that $|B| \geq 2$. If $b_2 \sim x$ for all $x \in B$, then $G \in \mathcal{F}$, and by Theorem 4.8, graph G has no convex $(2, nt)$-cover. Conversely, graph G has a convex $(2, nt)$-cover

$$\mathcal{P}_{2,nt}(G) = \{S_1 = d - conv(\{b_1, b_2\}), S_2 = B \cup \{b_1\}\}.$$

The theorem is proved. □

Algorithm 4.1 *Determines whether a graph belongs to the class* $\mathcal{F}$

Input: *Connected graph* $G = (X, U)$.

Output: *YES:* G *belongs to* $\mathcal{F}$*, or NO:* G *does not belong to* $\mathcal{F}$.

Step 1. *If* $|X| \leq 3$, *then return NO.*

Step 2. *If* $|X| = 4$, *then check whether* $G = C_4$. *If* $G = C_4$, *then return YES; otherwise, return NO.*

Step 3. *Check whether there exists or not a unique vertex* $x \in X$ *such that* $\Gamma(x) = \{y, z\}$ *and* y *is not adjacent to* z. *If not, then return NO.*

Step 4. *Check whether both* $X\backslash\{x, z\}$ *and* $X\backslash\{x, y\}$ *are cliques in* G. *If so, then return YES; otherwise, return NO.*

Theorem 4.10. *Algorithm 4.1 determines in* $O(n^2)$ *time whether a graph* G *belongs to the class* $\mathcal{F}$.

Proof. Evidently, steps 1 and 2 run in constant time. Step 3 executes in $O(n)$ time. It is clear that it can be verified in $O(n^2)$ time whether a given subgraph is a clique or not. Hence, step 4 operates in $O(n^2)$ time. Therefore, the execution time of Algorithm 4.1 is $O(n^2)$. □

4.3 Classes $\mathcal{J}$ and $\mathcal{H}$

Let us remark that every connected graph that contains simplicial vertices has at least two different convex $(2, t)$-covers. This follows directly from Theorem 4.1.

Now, we define two additional classes of graphs.

By $\mathscr{J}$ we denote a class of connected graphs on $n \geq 5$ vertices, which have at least two different convex $(2,t)$-covers and do not belong to $\mathscr{F}$.

By $\mathscr{H}$ we denote a class of connected graphs on $n \geq 5$ vertices, which have exactly one convex $(2,t)$-cover.

Theorem 4.11. *A graph $G \in \mathscr{J}$ has a convex $(2,nt)$-cover.*

Theorem 4.11 follows directly from Theorems 4.1 – 4.9.

Let $\mathscr{H}'$ be a subclass of $\mathscr{H}$ with the following properties:

a) $A \cap B = \emptyset$, where $A = \Gamma(x)\backslash\{y\}$, $B = \Gamma(y)\backslash\{x\}$ and $\{x,y\}$ is the trivial set of a convex $(2,t)$-cover of a graph;

b) For each $a \in A$, there exists $b \in B$ such that $b \sim a$, and for each $b \in B$, there exists $a \in A$, $a \sim b$;

c) $d - conv(A \cup B) = S_{nt}$, where S_{nt} is the nontrivial set of a convex $(2,t)$-cover of a graph;

d) $S_{nt} \neq A \cup B$. This implies that there exist $a \in A$, $b \in B$, $c \in C$ such that $d(a,b) = 2$ and $c \in \langle a,b \rangle$, where $C = S_{nt}\backslash(A \cup B)$.

Let $\mathscr{H}'' = \mathscr{H}\backslash\mathscr{H}'$.

Theorem 4.12. *A graph $G \in \mathscr{H}''$ has a convex $(2,nt)$-cover.*

Proof. Let $\mathcal{P}_{2,t}(G) = \{S_t = \{x,y\}, S_{nt}\}$ be a convex $(2,t)$-cover of G. We denote $A = \Gamma(x)\backslash\{y\}$, $B = \Gamma(y)\backslash\{x\}$. Since $G \in \mathscr{H}''$, we have $G \notin \mathscr{H}'$, and it follows that at least one property that characterizes the class $\mathscr{H}'$ is not satisfied.

If $A \cap B \neq \emptyset$, then G has a convex $(2,nt)$-cover

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{x,y,z\}, S_2 = S_{nt}\},$$

where $z \in A \cap B$.

Assume that property a) is satisfied. Conversely, by the above, G has a convex $(2,nt)$-cover. If there exists a vertex $a \in A$ for which does not exist $b \in B$ such that $b \sim a$, then G has a convex a $(2,nt)$-cover

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{x,y,a\}, S_2 = S_{nt}\}.$$

In the same way, if there exists $b \in B$ for which does not exist $a \in A$ such that $a \sim b$, then G has a convex a $(2,nt)$-cover

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{x, y, b\}, S_2 = S_{nt}\}.$$

If $d - conv(A \cup B) \neq S_{nt}$, then G has a convex $(2, nt)$-cover

$$\mathcal{P}_{2,nt}(G) = \{S_1 = \{x, y\} \cup d - conv(A \cup B), S_2 = S_{nt}\}.$$

If $S_{nt} = A \cup B$. Then we consider two cases.

1) Suppose $|A| \geq 2$ and $|B| \geq 2$. Then graph G has a convex $(2, nt)$-cover

$$\mathcal{P}_{2,nt}(G) = \{S_1 = A \cup \{x\}, S_2 = B \cup \{y\}\}.$$

2) Suppose that $|A| = 1$. Let v be a vertex of A. Since every graph of the class $\mathcal{H}''$ has at least five vertices, we get the inequality $|B| \geq 2$. Assume that properties a) and b) are satisfied. Conversely, by the above, graph G has a convex $(2, nt)$-cover. According to property b), vertex v is adjacent to all vertices of B and further $G \in \mathcal{F}$. By definition, $\mathcal{H}''$ is the class of graphs that has exactly one convex $(2, t)$-cover, but every graph that belongs to $\mathcal{F}$ has exactly two convex $(2, t)$-covers. This fact implies a contradiction. Similarly, if we suppose that $|B| = 1$, we also obtain a contradiction. So, $|A| \geq 2$ and $|B| \geq 2$, but in this case, G has a convex $(2, nt)$-cover. □

Algorithm 4.2. *Determines whether a graph belongs to one of the classes:* $\mathcal{G}, \mathcal{H}', \mathcal{H}''$.

Input: *Connected graph* $G = (X, U)$.

Output: *F*$\mathcal{G}$*: G belongs to* $\mathcal{G}$*; or F*$\mathcal{H}'$*: G belongs to* $\mathcal{H}'$*; or F*$\mathcal{H}''$*: G belongs to* $\mathcal{H}''$*; or NO: G does not belong to any of these classes.*

Step 1. *If* $|X| \leq 4$, *then return NO.*

Step 2. *Apply Algorithm 4.1. If Algorithm 4.1 returns YES, then return NO.*

Step 3. *Check whether there exists or not a simplicial vertex in* G. *If there is a simplicial vertex in* G, *then return F*$\mathcal{G}$.

Step 4. *Search all convex* $(2, t)$*-covers of* G, *i.e., define* $\widetilde{\mathcal{P}}_{2,t}(G)$. *For this purpose, search for all adjacent vertices* $x, y \in X$, *which satisfy the equality* $d - conv(X \backslash \{x, y\}) = X \backslash \{x, y\}$.

Step 5. *If* $\widetilde{\mathcal{P}}_{2,t}(G) = \emptyset$, *then return NO.*

Step 6. *If* $|\widetilde{\mathcal{P}}_{2,t}(G)| \geq 2$, *then return* $F\mathcal{J}$.

Step 7. *If* $A \cap B \neq \emptyset$, *where* $A = \Gamma(x)\backslash\{y\}$, $B = \Gamma(y)\backslash\{x\}$ *and* $\{x, y\}$ *is the trivial set of the single convex* $(2,t)$*-cover of* $\widetilde{\mathcal{P}}_{2,t}(G)$, *then return* $F\mathcal{H}''$.

Step 8. *Check whether there exists a vertex* $a \in A$ *such that for all vertices* $b \in B$, a *is not adjacent to* b, *or there exists a vertex* $b \in B$ *such that for all vertices* $a \in A$, b *is not adjacent to* a. *If there exists such a vertex* $a \in A$ *or* $b \in B$, *then return* $F\mathcal{H}''$.

Step 9. *Compute* $d - conv(A \cup B)$. *If* $d - conv(A \cup B) \neq S_{nt}$, *where* S_{nt} *is the nontrivial convex set of the single convex* $(2,t)$*-cover of* $\widetilde{\mathcal{P}}_{2,t}(G)$, *then return* $F\mathcal{H}''$.

Step 10. *If* $S_{nt} = A \cup B$, *then return* $F\mathcal{H}''$.

Step 11. *Return* $F\mathcal{H}'$.

Theorem 4.13. *Algorithm 4.2 determines in* $O(nm^2)$ *time whether a graph* G *belongs to one of the classes:* $\mathcal{J}, \mathcal{H}', \mathcal{H}''$.

Proof. It can be easily determined if a given graph G has at least five vertices. Since the complexity of Algorithm 4.1 is $O(n^2)$, it results that the complexity of step 2 is $O(n^2)$.

A vertex $x \in X$ is simplicial in graph G if and only if its neighborhood $\Gamma(x)$ is a clique, but determining if a given subset is a clique can be done in $O(n^2)$ time. Further, for checking whether there exists a simplicial vertex, it is needed $O(n^3)$ time. Therefore, the total complexity of step 3 is $O(n^3)$.

By Theorem 3.3, the convex hull of a subset $S \subseteq X(G)$ can be computed in $O(|d - conv(S)|m)$ time. Since $|d - conv(S)|$ can reach a value of n, we obtain that the complexity of step 9 is $O(nm)$.

The family $\widetilde{\mathcal{P}}_{2,t}(G)$ is obtained by applying the convex hull algorithm to set $X\backslash\{x, y\}$ for all pairs of adjacent vertices $x, y \in X$. The complexity of step 4 is $O(nm^2)$.

Clearly, steps 1, 5, 6 and 11 run in constant time, steps 7 and 10 run in $O(n)$ time, but step 8 executes in $O(n^2)$. As a result, we can decide in $O(nm^2)$ time whether or not a graph G belongs to one of the following classes: $\mathcal{J}, \mathcal{H}', \mathcal{H}''$. □

Theorem 4.14. *Let* $G = (X, U) \in \mathcal{H}'$ *be a graph that has a convex* $(2, t)$*-cover* $\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X \backslash \{x, y\}\}$ *and a convex* $(2, nt)$*-cover. Then* G *has a convex* $(2, nt)$*-cover* $\mathcal{P}_{2,nt}(G) = \{S_1, S_2\}$ *such that exactly one of the following conditions holds:*

a) $x, y \in S_1$ *and* $S_2 = X \backslash \{x, y\}$*;*

b) $x \in S_1$, $x \notin S_2$ *and* $y \in S_2$, $y \notin S_1$.

Proof. Let $\mathcal{P}'_{2,nt}(G) = \{S'_1, S'_2\}$ be a convex $(2, nt)$-cover of graph G. Suppose $x, y \in S'_1$. Then, $S_1 = S'_1$ and $S_2 = S_{nt}$. So, condition a) is satisfied. Otherwise, condition b) is satisfied. □

Theorem 4.15. *It can be decided in time* $O(n^2 m)$ *whether a graph* $G = (X, U) \in \mathcal{H}'$ *has a convex* $(2, nt)$*-cover that satisfies condition a) of Theorem 4.14 For this purpose, it is sufficient to determine whether there exists a vertex* $z \in A \cup B$ *such that* $S_{nt} \not\subseteq d - conv(\{x, y, z\})$*, where* $\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X \backslash \{x, y\}\}$ *is a convex* $(2, t)$*-cover of* G *and* $A = \Gamma(x) \backslash \{y\}$, $B = \Gamma(y) \backslash \{x\}$.

Proof. By the definition of $\mathcal{H}'$, graph G has no simplicial vertices and $|X| \geq 5$. Let $\mathcal{P}^1_{2,nt}(G) = \{S^1_1, S^1_2 = S_{nt}\}$ be a convex $(2, nt)$-cover of G and $x, y \in S^1_1$.

It is clear that there exists a vertex $z \in A \cup B$ such that $d - conv(\{x, y, z\}) \subseteq S^1_1$. Furthermore, G has a convex $(2, nt)$-cover

$$\mathcal{P}^2_{2,nt}(G) = \{S^2_1 = d - conv(x, y, z), S^2_2 = S_{nt}\}.$$

Therefore, it is sufficient to determine whether there exists a vertex z of the set $A \cup B$ such that $S_{nt} \not\subseteq d - conv(\{x, y, z\})$. For this purpose, we compute the convex hull of the set $\{x, y, z\}$ for each $z \in A \cup B$. If there is at least one vertex $z \in A \cup B$ such that $S_{nt} \not\subseteq d - conv(\{x, y, z\})$, then graph G has a convex $(2, nt)$-cover that satisfies condition a) of Theorem 4.14.

By Theorem 3.3, the computation of the convex hull of a set $S \subseteq X$ can be done in $O(|d - conv(S)|m)$ time. The decision whether G has a convex $(2, nt)$-cover that satisfies condition a) of Theorem 4.14 can be obtained by applying the convex hull algorithm at most $|A \cup B|$ times. Thus, the overall complexity is $O(n^2 m)$. □

Theorem 4.16. *Let* $G = (X, U) \in \mathcal{H}'$ *be a graph that has a convex* $(2, t)$*-cover* $\mathcal{P}_{2,t}(G) = \{S_t = \{x, y\}, S_{nt} = X \backslash \{x, y\}\}$ *and has no convex* $(2, nt)$*-cover that satisfies condition a) of Theorem 4.14, but has a convex* $(2, nt)$*-cover* $\mathcal{P}_{2,nt}(G) = \{S_1, S_2\}$ *that satisfies condition b) of Theorem 4.14, i.e.,* $x \in S_1$, $x \notin S_2$ *and* $y \in S_2$, $y \notin S_1$. *Then the following conditions hold:*

a) $(\Gamma(x) \backslash \{y\}) \subseteq S_1$ *and* $(\Gamma(x) \backslash \{y\}) \cap S_2 = \emptyset$*;*

b) $(\Gamma(y) \backslash \{x\}) \subseteq S_2$ *and* $(\Gamma(y) \backslash \{x\}) \cap S_1 = \emptyset$.

Proof. Assume that $(\Gamma(x) \backslash \{y\}) \cap S_2 \neq \emptyset$ or $(\Gamma(x) \backslash \{y\}) \not\subseteq S_1$, in other words, $(\Gamma(x) \backslash y) \cap S_2 \neq \emptyset$. Whence, we get $x \in S_2$. Since $x \in S_1$ and $y \in S_2$, $\mathcal{P}_{2,nt}(G)$ does not satisfy condition b) of Theorem 4.14. We have a contradiction. As above, if we assume that $(\Gamma(y) \backslash \{x\}) \cap S_1 \neq \emptyset$, or $(\Gamma(y) \backslash \{x\}) \not\subseteq S_2$, then we also get a contradiction. □

Generally, knowing all convex $(2, t)$-covers of a graph G does not essentially facilitate the task of determining whether G has a convex $(2, nt)$-cover. But it is crucial to know whether a graph that has convex $(2, t)$-covers also has a convex $(2, nt)$-cover.

We showed above that all graphs of the classes $\mathcal{J}$ and $\mathcal{H}''$ have a convex $(2, nt)$-cover, and none graph of $\mathcal{F}$ has a convex $(2, nt)$-cover. Also, we proved that it can be determined in polynomial time whether a graph belongs to one of the classes: $\mathcal{F}, \mathcal{J}, \mathcal{H}', \mathcal{H}''$.

We denote by $\mathcal{H}'(2, nt)$ the problem of deciding whether a graph $G \in \mathcal{H}'$ has a convex $(2, nt)$-cover. Now, we show the NP-completeness of the $\mathcal{H}'(2, nt)$ problem by reduction from the 1-IN-3 3SAT problem (the definition of this problem is given in section 3.1).

Theorem 4.17. *The* $\mathcal{H}'(2, nt)$ *problem is NP-complete.*

Proof. The $\mathcal{H}'(2, nt)$ problem is in NP because, by Theorem 3.3, verifying if a set is convex can be done in polynomial time.

We reduce 1-IN-3 3SAT problem to $\mathcal{H}'(2, nt)$ problem. We construct a particular graph $G = (X, U) \in \mathcal{H}'$ of $\mathcal{H}'(2, nt)$ problem from a generic instance $(V, \mathcal{C})$ of 1-IN-3 3SAT problem.

The vertex set X of G consists of the following sets:

a) vertices y and z;

b) $\mathcal{V} = \{\omega_1, \omega_2, \dots, \omega_n\}$, $Y = \{y_1, y_2, y_3, y_4\}$, $Y' = \{f, y_5, y_6, y_7, y_8, y_9\}$, $Z = \{z_1, z_2, z_3, z_4\}$, $Z' = \{t, z_5, z_6, z_7, z_8, z_9\}$;

c) $F = \{f_j : 1 \le j \le m\}$, $T = \{t_j : 1 \le j \le m\}$;

d) $L = \{l_j^i : 1 \le j \le m, 1 \le i \le 3\}$, $\mathcal{L} = \{\ell_j^i : 1 \le j \le m, 1 \le i \le 3\}$, $Q = \{q_j^i : 1 \le j \le m, 1 \le i \le 3\}$.

So, we get $X = \{y, z\} \cup \mathcal{V} \cup Y \cup Y' \cup Z \cup Z' \cup F \cup T \cup L \cup Q \cup \mathcal{L}$. Every variable $v_i \in V$ corresponds to a vertex $\omega_i \in \mathcal{V}$. Every clause $c_j \in \mathcal{C}$ corresponds to eleven vertices: $f_j, l_j^1, l_j^2, l_j^3, \ell_j^1, \ell_j^2, \ell_j^3, q_j^1, q_j^2, q_j^3, t_j$.

The edge set U of G satisfies the following conditions:

a) $y \sim z$, $y_4 \sim z_k$ and $z_4 \sim y_k$ for every $k, 1 \le k \le 4$;

b) $\mathcal{V} \cup Q$, $Y \cup \{y\}$ and $Z \cup \{z\}$ are cliques in G;

c) $\Gamma(f) = \mathcal{V} \cup Q \cup F \cup Y \cup \{y_6, y_7\}$ and $\Gamma(t) = \mathcal{V} \cup Q \cup T \cup Z \cup \cup \{z_6, z_7\}$;

d) $\Gamma(y_5) = F \cup Y \cup \{y_6, y_7\}$, $\Gamma(y_6) = Y \cup \{f, y_5, y_8, y_9, z_1\}$, $\Gamma(y_7) = Y \cup \{f, y_5, y_8, y_9, z_2\}$, $\Gamma(z_5) = T \cup Z \cup \{z_6, z_7\}$, $\Gamma(z_6) = Z \cup \{t, z_5, z_8, z_9, y_1\}$, $\Gamma(z_7) = Z \cup \{t, z_5, z_8, z_9, y_2\}$;

e) every clause $c_j = \{v_a, v_b, v_c\}$, $1 \le j \le m$, corresponds to eighteen edges: (l_j^1, ω_a), (l_j^2, ω_b), (l_j^3, ω_c), (l_j^1, f_j), (l_j^2, f_j), (l_j^3, f_j), (ℓ_j^1, t_j), (ℓ_j^2, t_j), (ℓ_j^3, t_j), (q_j^1, ℓ_j^1), (q_j^2, ℓ_j^2), (q_j^3, ℓ_j^3), (l_j^1, ℓ_j^2), (l_j^1, ℓ_j^3), (l_j^2, ℓ_j^1), (l_j^2, ℓ_j^3), (l_j^3, ℓ_j^1), (l_j^3, ℓ_j^2).

It is easy to see that G can be constructed in polynomial time.

Without loss of generality, we skip the trivial case $|\mathcal{C}| = 1$ of 1-IN-3 3SAT problem. We consider that $|\mathcal{C}| \ge 2$.

Firstly, we show that the obtained graph $G = (X, U)$ belongs to $\mathcal{H}$. Let us remember that $\mathcal{H}$ is a family of connected graphs on $n \ge 5$, which have exactly one convex $(2, t)$-cover. According to Theorem 4.1, any graph from $\mathcal{H}$ has no simplicial vertices. Moreover, it follows easily from the construction of G that this graph has no simplicial vertices but contains only one pair of adjacent vertices $\{y, z\}$ that satisfies the condition of Theorem 4.3. This means that G has exactly one convex $(2, t)$-cover $\mathcal{P}_{2,t} = \{S_t = \{y, z\}, S_{nt} = X \backslash \{y, z\}\}$ and further G belongs to class $\mathcal{H}$.

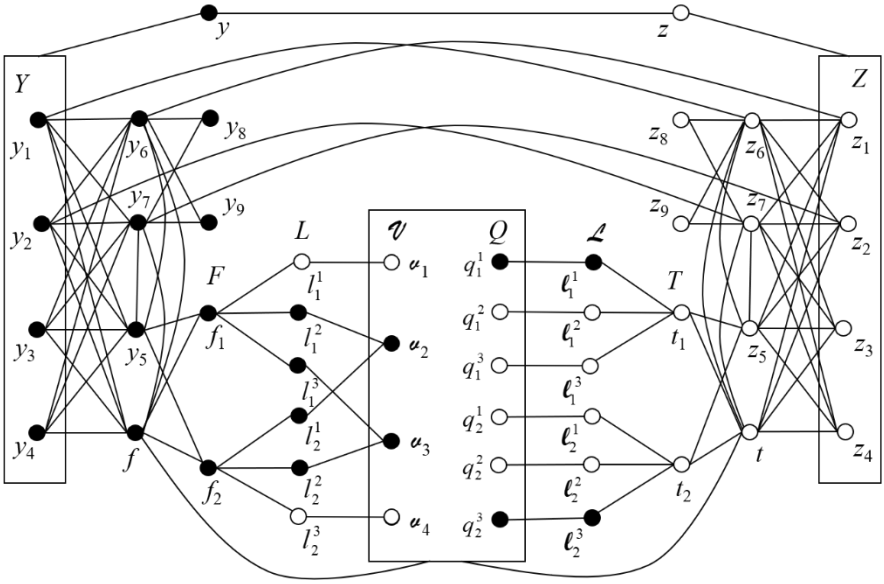


Figure 4.2. *The convex* $(2, nt)$*-cover of a graph* G *for the instance* $(V, \mathcal{C}) = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\})$. *The rectangles represent cliques*.

Secondly, we show that G is in $\mathcal{H}'$. To do this, we show that all properties that characterize the class $\mathcal{H}'$ hold. It is clear that properties a) and b) are satisfied. For sets $A = \Gamma(y)\backslash\{z\}$ and $B = \Gamma(z)\backslash\{y\}$, we can easily see that $d - conv(A \cup B) = S_{nt}$ and $S_{nt} \neq A \cup B$. Hence, properties c) and d) are also satisfied. So, G is in $\mathcal{H}'$.

Thirdly, we show that G has no convex $(2, nt)$-cover that satisfies the condition a) of Theorem 4.14. By the construction of G, we have $S_{nt} \subseteq d - conv(\{y, z, x\})$ for each $x \in A \cup B$. Further, considering Theorem 4.15, we obtain that G has no convex $(2, nt)$-cover that satisfies the condition a) of Theorem 4.14. Thus, if graph G has a convex $(2, nt)$-cover, then the condition b) of Theorems 4.14 and 4.16 should be satisfied.

Now, we prove that $\mathcal{C}$ is satisfiable if and only if G has a convex $(2, nt)$-cover.

If $G = (X, U)$ *has a convex* $(2, nt)$*-cover, then* $\mathcal{C}$ *is satisfiable.*

Let $\mathcal{P}_2(G) = \{S_f, S_t\}$ be a convex $(2, nt)$-cover of G where $y \in S_f$, $y \notin S_t$ and $z \in S_t$, $z \notin S_f$. We have $d - conv(\{y_i, z_j\}) = S_{nt} = X\backslash\{y, z\}$

for every $i, j \in \{8,9\}$. Further, $y_8, y_9 \in S_f$, $z_8, z_9 \in S_t$. We define two sets $S_1 = Y \cup Y' \cup F$ and $S_2 = Z \cup Z' \cup T$.

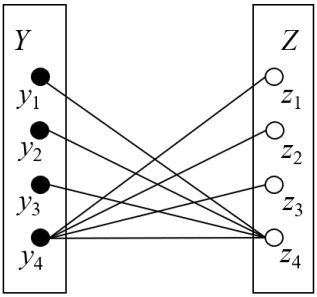


Figure 4.3. *Edges between Y and Z. The rectangles represent cliques.*

Let us distinguish some properties, which can be proved similarly as in the proof of Theorem 3.4:

Property 1: *$S_1 \cap S_t = \emptyset$ and $S_2 \cap S_f = \emptyset$.*

Property 2: *Sets $L, \mathcal{V}, \mathrm{Q}, \mathcal{L}$ are uniquely interdependent.*

Property 3: *For each j, $1 \le j \le m$, exactly one vertex of the set $L_j = \{l_j^1, l_j^2, l_j^3\}$ belongs to S_t and exactly one vertex of $\mathcal{L}_j = \{\ell_j^1, \ell_j^2, \ell_j^3\}$ belongs to S_f.*

We associate $\mathcal{V}$ with V and L with $\mathcal{C}$ such that a convex $(2, nt)$-cover represents a truth assignment for V, where the variable v_i is true iff $\omega_i \in S_t$.

Let us remark that sets S_f, S_t are nontrivial and disjoint. It follows from Properties 1, 2 and 3 that if G has a convex $(2, nt)$-cover $\mathcal{P}_2(G) = \{S_f, S_t\}$, then $\mathcal{C}$ is satisfiable.

If $\mathcal{C}$ is satisfiable, then $G = (X, U)$ has a convex $(2, nt)$-cover.

Suppose that there exists a truth assignment that satisfies $(V, \mathcal{C})$. We construct a convex $(2, nt)$-cover $\mathcal{P}_2(G) = \{S_f, S_t\}$ as follows:

Step 1. Define the set $S_t = Z \cup Z' \cup T \cup \{z\}$.

Step 2. For each true variable v_i of V, we add vertex ω_i and the set $L' = \Gamma(\omega_i) \cap L$ to S_t, and for each $l_j^a \in L'$, we add vertices q_j^b, ℓ_j^b to S_t such that $\ell_j^b \sim l_j^a$ and $q_j^b \sim \ell_j^b$.

Step 3. Define the set $S_f = X \backslash S_t$.

Clearly, for the resulting convex $(2, nt)$-cover $\mathcal{P}_2(G) = \{S_f, S_t\}$, the Properties 1, 2, and 3 are satisfied. Here sets S_f and S_t are nontrivial and disjoint. Hence, if $\mathcal{C}$ is satisfiable, then G has a convex $(2, nt)$-cover.

In Figure 4.2, we exhibit the graph G corresponding to a particular instance $(V, \mathcal{C}) = (\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\})$. Sets $Q \cup \mathcal{V} \cup \{f\}$, $Q \cup \mathcal{V} \cup \{t\}$, $Y \cup \{y\}$ and $Z \cup \{z\}$ are cliques in G. White vertices belong to S_t and black vertices belong to S_f. White vertices of $\mathcal{V}$ represent the variables of V set to true. All edges between Y and Z are represented in Figure 4.3, and all edges between L and $\mathcal{L}$ are represented in Figure 3.2. □

4.4 Nontrivial convex 2-covers of some classes of graphs

A graph G is *chordal* if every cycle of length four or more in G has a *chord*, which is an edge that is not part of the cycle but connects two vertices of the cycle.

Lemma 4.1. [20] *Each chordal graph has a simplicial vertex.*

Theorem 4.18. *A connected chordal graph G of order $n \geq 4$ has a nontrivial convex 2-cover.*

Proof. Note that any connected graph with $n = 4$ vertices, except for the cycle graph C_4, has a simplicial vertex. So by Theorems 2.7 and 4.2, G has a nontrivial convex 2-cover. □

Since complete graphs and trees are chordal graphs, we get the following result.

Corollary 4.2. *Let G be a complete graph or a tree of order $n \geq 4$. Then, G has a nontrivial convex 2-cover.*

A *power of cycle* C_n^k, $1 \leq k \leq n$, is a graph such that $X(C_n^k) = X(C_n)$ and $U(C_n^k) = \{(x_i, x_j) : x_i, x_j \in X(C_n^k), d_{C_n}(x_i, x_j) \leq k\}$.

Lemma 4.2. [3] *If $n \equiv 0,1,2 \pmod{2k}$, then a subset $S \subseteq X(C_n^k)$ formed by $\left\lceil \frac{n}{2} \right\rceil$ consecutive vertices of C_n^k is convex.*

Lemma 4.3. [3] *Let $S \subset X(C_n^k)$ be a non-clique convex set of graph C_n^k, $n > 2k + 2$ and $n \not\equiv 0,1,2 \pmod{2k}$. Then, $|S| < \left\lceil \frac{n}{2} \right\rceil$.*

Theorem 4.19. *A power of cycle C_n^k has a nontrivial convex 2-cover if and only if $n \geq 4$, $C_n^k \neq C_4$, and $n \leq 2k+2$ or $n \equiv 0,1,2 \pmod{2k}$.*

Proof. It is clear that C_n^k with $n < 4$ has no nontrivial convex 2-cover. We consider that $n \geq 4$.

Assume $n = 4$. By the definition of power of cycle, $1 \leq k \leq 2$. If $k = 1$, then $C_4^1 = C_4$ and it follows from Theorem 2.7 that C_n^k can not be covered by two nontrivial convex sets. If $k = 2$, then $C_4^2 = K_4$. In this case, by Corollary 4.2, C_4^2 has a nontrivial convex 2-cover.

Assume that $n \geq 5$. If $n \leq 2k+2$, then we see that any $k+1$ consecutive vertices form a clique and a nontrivial convex set. Hence, in this case, C_n^k has a nontrivial convex 2-cover. Also, Lemma 4.2 implies that C_n^k has a nontrivial convex 2-cover if $n \equiv 0,1,2 \pmod{2k}$. Moreover, from Lemma 4.3 it follows that C_n^k has no nontrivial convex 2-cover if $n > 2k+2$ and $n \not\equiv 0,1,2 \pmod{2k}$. □

Next, we examine the nontrivial convex 2-cover problem for cactus graphs. A *cactus* is a connected graph in which every edge belongs to at most one simple cycle.

Theorem 4.20. *A cactus G of order $n \geq 4$ has a nontrivial convex 2-cover if and only if $G \neq C_4$.*

Proof. Evidently, any cactus of order $n < 4$ has no nontrivial convex 2-cover.

Assume $n = 4$. Then by Theorem 2.7, there is a nontrivial convex 2-cover of G if and only if $G \neq C_4$.

Assume $n \geq 5$. If G has a simplicial vertex, then Theorem 4.2 yields that G has a nontrivial convex 2-cover. Now we consider that G has no simplicial vertices. If G is a cactus without a cut-vertex, then G is a cycle, and further, by Theorem 4.19, there exists a nontrivial convex 2-cover of G. Otherwise, there is a cut-vertex v of G. We remove v from G, and obtain a disconnected graph composed of $k \geq 2$ connected components $G_1, G_2, \dots, G_k$. Since there are no simplicial vertices in G, $|X(G_i)| \geq 2$ for each i, $1 \leq i \leq k$. Finally, graph G has the following nontrivial convex 2-cover:

$$\boldsymbol{\mathcal{P}}_{2,nt}(G) = \left\{\{v\} \cup X(G_1), \{v\} \cup \bigcup_{i=2}^{k} X(G_i)\right\}.$$

The theorem is proved. □

Theorem 4.21. *A disconnected graph G of order $n \geq 4$ has a nontrivial convex 2-cover.*

Proof. Let G be a disconnected graph composed of $k \geq 2$ connected components $G_1, G_2, \ldots, G_k$.

Assume $k \geq 4$. Then, G has a nontrivial convex 2-cover

$$\mathcal{P}_{2,nt}(G) = \left\{\bigcup_{i=1}^{3} X(G_i), \bigcup_{i=2}^{4} X(G_i)\right\}.$$

Assume that $k = 3$. There is at least one connected component containing at least two vertices because $n \geq 4$. Without loss of generality, let $|X(G_1)| \geq 2$. Then G has a nontrivial convex 2-cover

$$\mathcal{P}_{2,nt}(G) = \{X(G_1) \cup X(G_2), X(G_1) \cup X(G_3)\}.$$

Assume that $k = 2$. We consider three cases.

If $n = 4$ and each connected component consists of two adjacent vertices, then G has a nontrivial convex 2-cover

$$\mathcal{P}_{2,nt}(G) = \{\{x\} \cup X(G_2), \{y\} \cup X(G_2)\},$$

where $X(G_2) = \{x, y\}$.

If $n \geq 4$ and there is a connected component consisting of a single vertex (without loss of generality, let such a component be G_1), then G has a nontrivial convex 2-cover

$$\mathcal{P}_{2,nt}(G) = \{\{x, y\} \cup X(G_1), X(G_2)\},$$

where x and y are two adjacent vertices from G_2.

If $n \geq 5$, both connected components contain at least two vertices and one connected component consists of at least three vertices (let such a component be G_1), then G has a nontrivial convex 2-cover

$$\mathcal{P}_{2,nt}(G) = \{\{x\} \cup X(G_2), X(G_1)\},$$

where $x \in X(G_1)$. □

Chapter 5
Trees

In view of the fact that convex p-cover and convex p-partition problems of graphs are NP-complete (see Chapter 3), there is specific interest in studying these problems for different classes of graphs. In this chapter, we focus our attention on trees.

Let $L = [x^1, x^2, \dots, x^k]$ be a vertex path of a tree G. By $R_L(x)$ we denote the set of vertices $v \in X(G)$ for which there is a path $L' = [x, \dots, v]$ such that L' has no elements of L, except x, where $x \in L$.

Theorem 5.1. *A tree G with $n \geq 2$ vertices has a convex p-partition for every p, $2 \leq p \leq n$.*

Proof. It is obvious that G has at least two terminal vertices. Let x be a terminal vertex of G and let G' be the subgraph of G induced by the set of vertices $S = X(G)\backslash\{x\}$. Since G' is a tree and x is adjacent only to one vertex of S, sets $\{x\}$ and S are convex in G. So, the graph G has a convex 2-partition. If $|S| \geq 2$, we can further choose a terminal vertex y of G' and define a subtree G'' of G' induced by the set of vertices $S' = X(G')\backslash\{y\}$. Note that the sets $\{x\}$, $\{y\}$ and S' form a convex 3-partition of G. If $|S'| \geq 2$, then in the same vein, we can obtain a partition of G into 4, 5, …, and n convex sets. □

Bearing in mind that each convex p-partition is a convex p-cover of a graph, we get the following result.

Corollary 5.1. *A tree G with $n \geq 2$ vertices has a convex p-cover for every p, $2 \leq p \leq n$.*

5.1 Nontrivial convex covers of trees

The nontrivial convex p-cover problem of trees is not as simple as the convex p-cover problem. Therefore, we will carefully consider nontrivial convex covers of trees.

Lemma 5.1. *A tree G with $n \geq 4$ vertices has a nontrivial convex 2-cover.*

Proof. It is clear that $diam(G) \geq 2$. Let x and y be two vertices of G such that $d(x, y) = diam(G)$. Then x and y are both terminal vertices of G, and it can be easily checked that G can be covered by two nontrivial convex sets $X(G)\backslash\{x\}$ and $X(G)\backslash\{y\}$. □

Theorem 5.2. *Let G be a tree with $diam(G) \geq 3$. There exists a maximum nontrivial convex cover $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ such that every terminal vertex of G is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ and any two terminal vertices do not belong to the same set of $\mathcal{P}_{\varphi_{cn}^{max}}(G)$.*

Proof. We know from Lemma 5.1 that G has a nontrivial convex cover. Let $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ be a maximum nontrivial convex cover of G, where there is at least one terminal vertex x that is non-resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. Since x is a terminal vertex of G and $diam(G) \geq 3$, there is a vertex y adjacent to x, which is adjacent to the set of nonterminal vertices S, $S \neq \emptyset$, and to the set of terminal vertices S', $S' \neq \emptyset$, of G. Note that y is also non-resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$.

We consider two cases.

1) Suppose that S contains a vertex z that is non-resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. Firstly, we replace vertex x by vertex z in every set of $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ that contains x. Secondly, we add a convex set $\{x, y, z\}$ to $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. Further, we get a nontrivial convex cover $\mathcal{P}(G)$ in which x is a resident vertex and $|\mathcal{P}(G)| > |\mathcal{P}_{\varphi_{cn}^{max}}(G)|$. Hence, we get a contradiction.

2) Now suppose that every vertex of S is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. Firstly, we choose a vertex z of S and a set Z of $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ that contains z. Secondly, we replace x by z in every set of $\mathcal{P}_{\varphi_{cn}^{max}}(G)\backslash\{Z\}$ which contains x. After, we add x and y to Z, and obtain a new nontrivial convex cover $\mathcal{P}(G)$, $|\mathcal{P}(G)| = |\mathcal{P}_{\varphi_{cn}^{max}}(G)|$ in which x is resident. On the other hand, if

now the set S' contains one more vertex that is non-resident in $\mathcal{P}(G)$, then taking into account case 1) we again obtain a contradiction.

Consequently, there exists a maximum nontrivial convex cover $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ of graph G such that every terminal vertex of G is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$.

Now suppose that there are at least two terminal vertices x and y which belong to the same nontrivial convex set S of $\mathcal{P}_{\varphi_{cn}^{max}}(G)$.

Let us consider two cases.

1) Assume that $|S| \geq 4$. In this case, we replace S in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ by two convex sets $S' = S\backslash\{x\}$, $|S'| \geq 3$, and $S'' = S\backslash\{y\}$, $|S''| \geq 3$. Further, we obtain a new nontrivial convex cover $\mathcal{P}(G)$, $|\mathcal{P}(G)| > |\mathcal{P}_{\varphi_{cn}^{max}}(G)|$. So, we have a contradiction.

2) Assume that $|S| = 3$. Here $S = \{x, y, z\}$ and $\Gamma(x) = \Gamma(y) = \{z\}$. As above, note that the set $\Gamma(z)\backslash\{x, y\}$ contains at least one nonterminal vertex h.

If h is non-resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$, then we replace S by two convex sets $\{x, z, h\}$, $\{y, z, h\}$, and obtain a new nontrivial convex cover $\mathcal{P}(G)$, $|\mathcal{P}(G)| > |\mathcal{P}_{\varphi_{cn}^{max}}(G)|$, in which vertices x, y belong to different sets. Consequently, we get a contradiction.

If all nonterminal vertices of $\Gamma(z)\backslash\{x, y\}$ are resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$, then we choose a set H that contains h. Further, we subtract x from S and add it to H. Also, we add h to S and z to H. Hence, we obtain a new nontrivial convex cover $\mathcal{P}(G)$ of G in which vertices x, y belong to different sets and $|\mathcal{P}(G)| = |\mathcal{P}_{\varphi_{cn}^{max}}(G)|$.

It follows that there exists a maximum nontrivial convex cover $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ such that any two terminal vertices of G do not belong to the same set of $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. □

As a consequence of Theorem 5.2, we obtain Corollary 5.2.

Corollary 5.2. *Let G be a tree with $diam(G) \geq 3$ and p terminal vertices. Then, $\varphi_{cn}^{max}(G) \geq p$.*

Corollary 5.3. *Let G be a tree with $diam(G) \geq 3$ and p terminal vertices, where every nonterminal vertex of G is adjacent to at least one terminal vertex. Then, $\varphi_{cn}^{max}(G) = p$*

Corollary 5.4. *Let G be a tree with $3 \leq diam(G) \leq 5$ and p terminal vertices. Then, $\varphi_{cn}^{max}(G) = p$.*

The proof of Corollaries 5.3 and 5.4 easily derives from Theorem 5.2 and from the fact that all nonterminal vertices of G become covered by nontrivial convex sets, which cover all terminal vertices of G.

Theorem 5.3. *A tree G on $n \geq 4$ vertices has a nontrivial convex p-cover for every p, $2 \leq p \leq \varphi_{cn}^{max}(G)$.*

Proof. Let G be a tree on $n \geq 4$ vertices. Considering Lemma 5.1, G can be covered by two nontrivial convex sets. Let $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ be a maximum nontrivial convex cover of G. If $\varphi_{cn}^{max}(G) = 2$, then the theorem is proved. Let us analyze the case when $\varphi_{cn}^{max}(G) \geq 3$. We use the following procedure. We select sets X_1 and X_2 of $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ such that there are two adjacent vertices $x_1 \in X_1$, $x_2 \in X_2$. Since the set $S = X_1 \cup X_2$ is nontrivial and convex in G, by excluding from $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ sets X_1 and X_2, and then by adding S, we obtain a new nontrivial convex cover $\mathcal{P}(G)$ that covers G by $p = \varphi_{cn}^{max}(G) - 1$ nontrivial and convex sets. If $p = 2$, then the theorem is correct. Otherwise, if $p \geq 3$, then by repeating this procedure $\varphi_{cn}^{max}(G) - 3$ times for $\mathcal{P}(G)$, we get a nontrivial convex 2-cover of G. □

We denote by $C(G)$, $|C(G)| = p$, a set of all terminal vertices of a tree G. By $M(G)$, where $|M(G)| = q$, we denote a set of all vertices x of G for which the distance between any vertex of $C(G)$ and x is greater than or equal to 3, and exists a vertex $c \in C(G)$, $d(x, c) = 3$.

Theorem 5.4. *Let G be a tree with $diam(G) \geq 6$ and $M(G) \neq \emptyset$. Then $\varphi_{cn}^{max}(G) \geq p + q$.*

Proof. We define a family of nontrivial convex sets $\mathcal{P}(G) = \emptyset$ that will cover G. For every terminal vertex $c \in C(G)$ we select the nearest vertex $x \in M(G)$ and the corresponding path $L = [x, x_1, x_2, \dots, x_k, c]$, $k \geq 2$. Since $S_c = \{x_1, x_2, \dots, x_k, c\}$ is a nontrivial and convex set, we add S_c to $\mathcal{P}(G)$. Besides, for every $x \in M(G)$ we select a terminal vertex $c \in C(G)$, $d(x, c) = 3$, and for the obtained path L with $k = 2$, we form a nontrivial convex set $S_x = \{x, x_1, x_2\}$ and add it to $\mathcal{P}(G)$. If there remain some uncovered vertices, then we select an uncovered vertex y that is adjacent to a vertex $z \in S$, $S \in \mathcal{P}(G)$, and further extend S by adding y to it. We see

that now every vertex of $A = C(G) \cup M(G)$ is resident in $\mathcal{P}(G)$, and any two vertices from A do not belong to the same set of $\mathcal{P}(G)$. Consequently, we obtain a nontrivial convex cover $\mathcal{P}(G)$ such that $|\mathcal{P}(G)| = p + q$. So, $\varphi_{cn}^{max}(G) \geq p + q$. □

The following theorem gives a significant result.

Theorem 5.5. *If a tree G has $n \geq 4$ vertices, then there exists a maximum nontrivial convex cover $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ such that every set $S \in \mathcal{P}_{\varphi_{cn}^{max}}(G)$ contains a path $L = [x, y, z]$, where x is a resident vertex in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$.*

Proof. If $diam(G) = 2$, then the theorem is true. If $3 \leq diam(G) \leq 5$, then it follows from Corollary 5.4 that the theorem is true. If $diam(G) \geq 6$, then taking into account Theorem 5.2, there is a family $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ in which for every terminal vertex $x \in C(G)$ that is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ and for a set $S \in \mathcal{P}_{\varphi_{cn}^{max}}(G)$, $x \in S$, there exists a path $L = [x, y, z]$, where $y, z \in S$.

We define a family of nontrivial convex sets $\mathcal{P}(G) = \emptyset$ that will cover G and a set of vertices $D = \emptyset$. If there is a set $A \in \mathcal{P}_{\varphi_{cn}^{max}}(G)$, containing a terminal vertex $a \in A$ that is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$, and there exists another set $B \in \mathcal{P}_{\varphi_{cn}^{max}}(G)$ such that $A \cap B \neq \emptyset$, $|A \backslash B| \geq 2$ or $|B \backslash A| \geq 2$, then we denote by B_r the set of all resident vertices of B in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. Next, we select a vertex $b \in B_r$ for which the distance $d(a, b)$ is maximum. Also, we denote by B_b the set of all vertices b' of B for which a path between b' and a contains b. Evidently, vertex b belongs to B_b. We now define two sets:

$$A' = A \cup (B \backslash B_b) \text{ and } B' = (A' \cup \{b\}) \backslash \{a\}.$$

It can easily be checked that A', B' are nontrivial convex sets, and there are paths $[a, c, d]$ and $[b, c', d']$, where $c, d \in A'$ and $c', d' \in B'$. Further, we replace sets A and B by A' and B' in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. If there still remains such a set $A \in \mathcal{P}_{\varphi_{cn}^{max}}(G)$ that satisfies the conditions mentioned above, then we repeat the described process. Otherwise, we define a family $\mathcal{V}$ that consists of sets from $\mathcal{P}_{\varphi_{cn}^{max}}(G)$, which contain exactly one terminal vertex that is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G)$. Next, we select a set $A \in \mathcal{V}$ and define a family $\mathcal{W}_A$, composed of sets that belong to $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ and intersect A. Let

$D_A = A \cup \bigcup_{B \in \mathcal{W}_A} B$. We add vertices of D_A to D, remove A and every set of $\mathcal{W}_A$ from $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ and from $\mathcal{V}$. Then add them to $\mathcal{P}(G)$. If $\mathcal{V} \neq \emptyset$, then we choose another set $A \in \mathcal{V}$ and repeat the above procedure. On the contrary case, we get a new graph G' by removing from G vertices of D and all edges incident to them. If $X(G') = \emptyset$, then $|\mathcal{P}(G)| = |\mathcal{P}_{\varphi_{cn}^{max}}(G)|$. This implies the correctness of the theorem.

If $X(G') \neq \emptyset$, then G' consists of $k \geq 1$ subtrees G_1, G_2, ..., G_k. It is clear that $X(G_i) \geq 3$, $1 \leq i \leq k$, and every set of $\mathcal{P}(G)$ does not intersect any set of $\mathcal{P}'(G)$, obtained from $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ after the elimination of convex sets in the actions described above. Therefore, we have

$$|\mathcal{P}'(G)| + |\mathcal{P}(G)| = \varphi_{cn}^{max}(G).$$

If $2 \leq diam(G_i) \leq 5$, for every i, $1 \leq i \leq k$, then, considering that if $|X(G_i)| = 3$ then $|\mathcal{P}(G_i)| = 1$ and by Corollary 5.4, a maximum nontrivial convex cover $\mathcal{P}(G_i)$ is easily obtained for every G_i such that the statement of the theorem holds. Conversely, for every subtree G_i, $diam(G_i) \geq 6$, we define a family of sets $\mathcal{P}(G_i) = \emptyset$ that will cover G_i, and then recursively fill it using rationales from the above. Further, we get

$$\sum_{i=1}^{k} |\mathcal{P}(G_i)| + |\mathcal{P}(G)| = \varphi_{cn}^{max}(G).$$

Finally, we add all sets from $\mathcal{P}(G_i)$, $1 \leq i \leq k$, to $\mathcal{P}(G)$ and see that the theorem is proved. □

Suppose that $diam(G) \geq 6$, then we define the set:

$$N(G) = X(G) \setminus \left(C(G) \cup \bigcup_{x \in C(G)} \Gamma(x) \right).$$

The set $N(G)$ is empty if and only if every nonterminal vertex of G is adjacent to at least one terminal vertex of G, but in this case, accordingly to Corollary 5.3, we obtain $\varphi_{cn}^{max}(G) = p$.

Assume further that $N(G) \neq \emptyset$. Let x be a vertex of $N(G)$. Since x is an articulation vertex, through the elimination of x from G we obtain $|\Gamma(x)|$ connected components G_x^y, $y \in \Gamma(x)$. For every vertex $y \in \Gamma(x)$, we get the family of subtrees:

$$\mathcal{V}_x^y(G) = \left\{ {}^*G_x^y \right\} \cup \{G_x^z : z \in \Gamma(x) \backslash \{y\}\},$$

where ${}^*G_x^y$ is a subtree of G obtained by adding the vertex x to G_x^y such that $x \sim y$.

Thus, we get the family of subfamilies of subtrees:

$$\mathcal{V}_x(G) = \left\{\mathcal{V}_x^y(G) : y \in \Gamma(x)\right\}.$$

For the sake of estimation of $\varphi_{cn}^{max}(G)$, we consider that if $0 \leq |X(G)| \leq 2$, then $\varphi_{cn}^{max}(G) = 0$. Similarly, if $|X(G)| = 3$, then we consider that $\varphi_{cn}^{max}(G) = 1$. We obtain the recursive formula for determining the value of $\varphi_{cn}^{max}(G)$, reflected in the following two theorems.

Let us remark that for a tree G with $diam(G) = 2$ and $|X(G)| \geq 4$, it can easily be checked that $\varphi_{cn}^{max}(G) = p - 1$. Therefore, taking into account the above considerations and Corollaries 5.3, 5.4, we obtain Theorem 5.6.

Theorem 5.6. *Let G be a tree. If $diam(G) \leq 5$ or $diam(G) \geq 6$ and $N(G) = \emptyset$, then the following relation holds*:

$$\varphi_{cn}^{max}(G) = \begin{cases} p, & if\ 3 \leq diam(G) \leq 5, \text{or} \\ & diam(G) \geq 6\ and\ N(G) = \emptyset \\ p - 1, & if\ diam(G) = 2 \\ 0, & if\ 0 \leq diam(G) \leq 1. \end{cases}$$

If $diam(T) \geq 6$ and $N(G) \neq \emptyset$, then by Theorem 5.5, we get Theorem 5.7.

Theorem 5.7. *Let G be a tree with $diam(G) \geq 6$ and $N(G) \neq \emptyset$, then the following relation holds*:

$$\varphi_{cn}^{max}(G) = \max\left\{p, \max_{x \in N(G)}\left\{\max_{y \in \Gamma(x)}\left\{\sum_{H \in \mathcal{V}_x^y(G)} \varphi_{cn}^{max}(H)\right\}\right\}\right\}.$$

Taking into account Theorems 5.6 and 5.7, we propose a recursive procedure $Max\varphi(G)$ that determines the maximum nontrivial convex cover number $\varphi_{cn}^{max}(G)$ of a tree G.

Procedure $Max\varphi(G)$. *Determines $\varphi_{cn}^{max}(G)$ for a tree G.*

Input: *A tree G.*

Output: $\varphi_{cn}^{max}(G)$.

Step 1. Compute $diam(G)$.

Step 2. If $0 \leq diam(G) \leq 1$, then return 0.

Step 3. Determine the set of all terminal vertices $C(G)$, $|C(G)| = p$.

Step 4. If $diam(G) = 2$, then return $p - 1$.

Step 5. Determine the set $N(G) = X(G)\backslash\big(C(G) \cup \cup_{x \in C(G)} \Gamma(x)\big)$.

Step 6. If $3 \leq diam(G) \leq 5$, or $diam(G) \geq 6$ and $N(G) = \emptyset$, then return p; otherwise, for every $x \in N(G)$ and $y \in \Gamma(x)$, apply procedure $Max\theta(H)$ for each $H \in \mathscr{V}_x^y(G)$ and, after all, calculate the sums $\sum_{H \in \mathscr{V}_x^y(G)} \varphi_{cn}^{max}(H)$. Finally, return

$$\varphi_{cn}^{max}(G) = \max\left\{p, \max_{x \in N(G)}\left\{\max_{y \in \Gamma(x)}\left\{\sum_{H \in \mathscr{V}_x^y(G)} \varphi_{cn}^{max}(H)\right\}\right\}\right\}.$$

We illustrate an example of the determination of the number $\varphi_{cn}^{\max}(G)$ for a tree G from Figure 5.1. We see that $diam(G) = 8 \geq 6$, and the set of terminal vertices $C(G) = \{x_1, x_2, x_3, x_6, x_{13}, x_{15}, x_{16}, x_{17}, x_{18}\}$ consists of 9 elements.

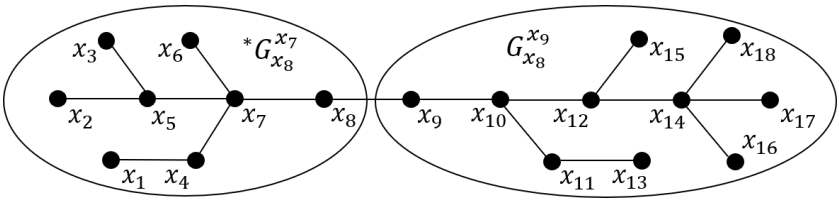


Figure 5.1. *The tree G and two subtrees ${}^*G_{x_8}^{x_7}$ and $G_{x_8}^{x_9}$.*

For G, we have

$$N(G) = X(G)\backslash\left(C(G) \cup \bigcup_{x \in C(G)} \Gamma(x)\right) = \{x_8, x_9, x_{10}\}.$$

In total, there are three families of subfamilies of subtrees:

$$\mathcal{V}_{x_8}(G) = \{\mathcal{V}_{x_8}^{x_7}(G), \mathcal{V}_{x_8}^{x_9}(G)\},$$

$$\mathcal{V}_{x_9}(G) = \{\mathcal{V}_{x_9}^{x_8}(G), \mathcal{V}_{x_9}^{x_{10}}(G)\},$$

$$\mathcal{V}_{x_{10}}(G) = \{\mathcal{V}_{x_{10}}^{x_9}(G), \mathcal{V}_{x_{10}}^{x_{11}}(G), \mathcal{V}_{x_{10}}^{x_{12}}(G)\}.$$

We examine each family of subfamilies in turn. We start with $\mathcal{V}_{x_8}(G)$ that consists of two families of subtrees:

$$\mathcal{V}_{x_8}^{x_7}(G) = \{ {}^*G_{x_8}^{x_7}, G_{x_8}^{x_9}\} \text{ and } \mathcal{V}_{x_8}^{x_9}(G) = \{G_{x_8}^{x_7}, {}^*G_{x_8}^{x_9}\}.$$

Note that, for each subtree H belonging to $\mathcal{V}_{x_8}^{x_7}(G)$ or $\mathcal{V}_{x_8}^{x_9}(G)$, the relation $diam(H) \leq 5$ holds. Therefore, we get:

$$\varphi_{cn}^{\max}({}^*G_{x_8}^{x_7}) = 5, \varphi_{cn}^{\max}(G_{x_8}^{x_7}) = 4,$$

$$\varphi_{cn}^{\max}({}^*G_{x_8}^{x_9}) = 6, \varphi_{cn}^{\max}(G_{x_8}^{x_9}) = 6.$$

We calculate the sums:

$$\sum\nolimits_{H \in \mathcal{V}_{x_8}^{x_7}(G)} \varphi_{cn}^{max}(H) = \varphi_{cn}^{max}({}^*G_{x_8}^{x_7}) + \varphi_{cn}^{max}(G_{x_8}^{x_9}) = 5 + 6 = 11,$$

$$\sum\nolimits_{H \in \mathcal{V}_{x_8}^{x_9}(G)} \varphi_{cn}^{max}(H) = \varphi_{cn}^{max}(G_{x_8}^{x_7}) + \varphi_{cn}^{max}({}^*G_{x_8}^{x_9}) = 4 + 6 = 10.$$

Now, we choose the maximum value

$$\mathrm{m}_1 = \max\left\{\sum\nolimits_{H \in \mathcal{V}_{x_8}^{x_7}(G)} \varphi_{cn}^{max}(H), \sum\nolimits_{H \in \mathcal{V}_{x_8}^{x_9}(G)} \varphi_{cn}^{max}(H)\right\} =$$

$$= \max\{11,10\} = 11.$$

The family $\mathcal{V}_{x_9}(G)$ consists of two families of subtrees:

$$\mathcal{V}_{x_9}^{x_8}(G) = \{ {}^*G_{x_9}^{x_8}, G_{x_9}^{x_{10}}\} \text{ and } \mathcal{V}_{x_9}^{x_{10}}(G) = \{G_{x_9}^{x_8}, {}^*G_{x_9}^{x_{10}}\}.$$

The diameter of each subtree H in these families is $diam(H) \leq 5$. Hence, we get:

$$\varphi_{cn}^{max}({}^*G_{x_9}^{x_8}) = 5, \varphi_{cn}^{max}(G_{x_9}^{x_8}) = 5,$$

$$\varphi_{cn}^{max}({}^*G_{x_9}^{x_{10}}) = 6, \varphi_{cn}^{max}(G_{x_9}^{x_{10}}) = 5.$$

We calculate the sums:

$$\sum_{H\in\mathscr{V}_{x_9}^{x_8}(G)}\varphi_{cn}^{max}(H)=\varphi_{cn}^{max}\left({}^{*}G_{x_9}^{x_8}\right)+\varphi_{cn}^{max}\left(G_{x_9}^{x_{10}}\right)=5+5=10,$$

$$\sum_{H\in\mathscr{V}_{x_9}^{x_{10}}(G)}\varphi_{cn}^{max}(H)=\varphi_{cn}^{max}\left(G_{x_9}^{x_8}\right)+\varphi_{cn}^{max}\left({}^{*}G_{x_9}^{x_{10}}\right)=5+6=11.$$

Now, we choose the maximum value

$$\mathrm{m}_2=\max\left\{\sum_{H\in\mathscr{V}_{x_9}^{x_8}(G)}\varphi_{cn}^{max}(H),\sum_{H\in\mathscr{V}_{x_9}^{x_{10}}(G)}\varphi_{cn}^{max}(H)\right\}=$$
$$=\max\{10,11\}=11.$$

Finally, we examine the family $\mathscr{V}_{x_{10}}(G)$ that consists of three families of subtrees:

$$\mathscr{V}_{x_{10}}^{x_9}(G)=\left\{{}^{*}G_{x_{10}}^{x_9},G_{x_{10}}^{x_{11}},G_{x_{10}}^{x_{12}}\right\},$$
$$\mathscr{V}_{x_{10}}^{x_{11}}(G)=\left\{G_{x_{10}}^{x_9},{}^{*}G_{x_{10}}^{x_{11}},G_{x_{10}}^{x_{12}}\right\},$$
$$\mathscr{V}_{x_{10}}^{x_{12}}(G)=\left\{G_{x_{10}}^{x_9},G_{x_{10}}^{x_{11}},{}^{*}G_{x_{10}}^{x_{12}}\right\}.$$

Since the diameter of each subtree in these families is less than 6, we obtain:

$$\varphi_{cn}^{max}\left({}^{*}G_{x_{10}}^{x_9}\right)=5,\ \varphi_{cn}^{max}\left(G_{x_{10}}^{x_9}\right)=5,$$
$$\varphi_{cn}^{max}\left({}^{*}G_{x_{10}}^{x_{11}}\right)=1,\ \varphi_{cn}^{max}\left(G_{x_{10}}^{x_{11}}\right)=0,$$
$$\varphi_{cn}^{max}\left({}^{*}G_{x_{10}}^{x_{12}}\right)=5,\ \varphi_{cn}^{max}\left(G_{x_{10}}^{x_{12}}\right)=4.$$

We calculate the sums:

$$\sum_{H\in\mathscr{V}_{x_{10}}^{x_9}(G)}\varphi_{cn}^{max}(H)=\varphi_{cn}^{max}\left({}^{*}G_{x_{10}}^{x_9}\right)+\varphi_{cn}^{max}\left(G_{x_{10}}^{x_{11}}\right)+\varphi_{cn}^{max}\left(G_{x_{10}}^{x_{12}}\right)$$
$$=5+0+4=9,$$

$$\sum_{H\in\mathscr{V}_{x_{10}}^{x_{11}}(G)}\varphi_{cn}^{max}(H)=\varphi_{cn}^{max}\left(G_{x_{10}}^{x_9}\right)+\varphi_{cn}^{max}\left({}^{*}G_{x_{10}}^{x_{11}}\right)+\varphi_{cn}^{max}\left(G_{x_{10}}^{x_{12}}\right)$$
$$=5+1+4=10,$$

$$\sum_{H\in\mathscr{V}_{x_{10}}^{x_{12}}(G)}\varphi_{cn}^{max}(H)=\varphi_{cn}^{max}\left(G_{x_{10}}^{x_9}\right)+\varphi_{cn}^{max}\left(G_{x_{10}}^{x_{11}}\right)+\varphi_{cn}^{max}\left({}^{*}G_{x_{10}}^{x_{12}}\right)$$
$$=5+0+5=10.$$

We choose the maximum value

$$\mathrm{m}_3 = \max\left\{\sum_{H\in\mathscr{V}_{x_{10}}^{x_9}(G)} \varphi_{cn}^{max}(H), \sum_{H\in\mathscr{V}_{x_{10}}^{x_{11}}(G)} \varphi_{cn}^{max}(H), \sum_{H\in\mathscr{V}_{x_{10}}^{x_{12}}(G)} \varphi_{cn}^{max}(H)\right\} =$$

$$= \max\{9,10,10\} = 10.$$

After all, we obtain the maximum nontrivial convex cover number of a tree G:

$$\varphi_{cn}^{max}(G) = \max\{|C(G)|, \max\{\mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3\}\} =$$

$$\max\{9, \max\{11,11,10\}\} = \max\{9,11\} = 11.$$

5.2 Nontrivial convex partitions of trees

Now, we have to analyze nontrivial convex partitions of trees. The following two classes of trees $\mathscr{A}$ and $\mathscr{B}$ are needed for the sequel.

$\mathscr{A}$ is a class of trees G, whose sets of vertices and sets of edges satisfy the below conditions:

1) $X(G) = \{x, y, x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_{k'}\}$, where $k, k' \geq 2$;

2) $U(G) = \{(x,y)\} \cup \bigcup_{i=1}^{k} \{(x,x_i)\} \cup \bigcup_{i=1}^{k'} \{(y,y_i)\}$.

$\mathscr{B}$ is a class of trees G, which are constructed as follows:

1) We choose $k \geq 0$, $k' \geq 2$, $k_1 \geq 2$ and for every i, $2 \leq i \leq k'$, we choose the value of $k_i \geq 1$;

2) If $k \geq 1$, then $X = \{x_0\} \cup \bigcup_{i=1}^{k} \{x_i\}$ and $U = \bigcup_{i=1}^{k} \{(x_0, x_i)\}$, otherwise $X = \{x_0\}$ and $U = \emptyset$;

3) We obtain the sets $X(G) = X \cup \bigcup_{i=1}^{k'} \bigcup_{j=0}^{k_i} \{x_i^j\}$ and $U(G) = U \cup \bigcup_{i=1}^{k'} \{(x_0, x_i^0)\} \cup \bigcup_{i=1}^{k'} \bigcup_{j=1}^{k_i} \{(x_i^0, x_i^j)\}$.

It can easily be checked that the diameter of each tree of $\mathscr{A}$ is 3, and the diameter of each tree of $\mathscr{B}$ is 4.

Moreover, every tree of $\mathscr{A}$ and every tree of $\mathscr{B}$ has at least 6 vertices. The general structure of trees of the described classes $\mathscr{A}$ and $\mathscr{B}$ is represented in Figure 5.2.

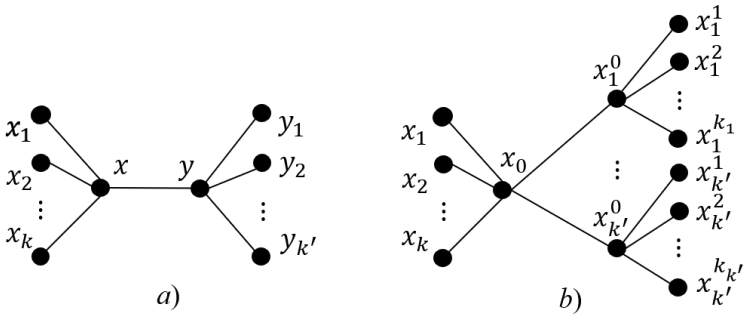


Figure 5.2. *The classes of trees* $\mathcal{A}$ (*case a*)) *and* $\mathcal{B}$ (*case b*)).

Algorithm 5.1. *Determines whether a tree belongs to one of the classes:* $\mathcal{A}$, $\mathcal{B}$.

Input: *A tree* G.

Output: *YES-*$\mathcal{A}$*: G belongs to* $\mathcal{A}$*; or YES-*$\mathcal{B}$*: G belongs to* $\mathcal{B}$*; or NO: G does not belong to either* $\mathcal{A}$ *or* $\mathcal{B}$.

Step 1. If $0 \leq |X(G)| \leq 5$, *then return NO.*

Step 2. Compute $diam(G)$. *If* $diam(G) \leq 2$ *or* $diam(G) \geq 5$, *then return NO; otherwise, if* $diam(G) = 4$, *then go to Step 4.*

Step 3. Choose vertices $x, y \in X(G)$, $x \neq y$, *such that* $|\Gamma(x)| \geq 2$, $|\Gamma(y)| \geq 2$. *If* $|\Gamma(x)| \geq 3$ *and* $|\Gamma(y)| \geq 3$, *then return YES–*$\mathcal{A}$*; otherwise, return NO.*

Step 4. Check if there exist two different terminal vertices $x, y \in X(G)$ *such that* $\Gamma(x) \cap \Gamma(y) \neq \emptyset$ *and there is a terminal vertex* $z \in X(G)$, $d(x, z) = diam(G)$. *If there exist such vertices* $x, y \in X(G)$, *then return YES–*$\mathcal{B}$*; otherwise, return NO.*

Theorem 5.8. *Algorithm 5.1 determines in* $O(n^3)$ *time whether a tree G belongs to one of the classes* $\mathcal{A}$ *or* $\mathcal{B}$.

Proof. Correctness of the algorithm results from the structure of trees of the classes $\mathcal{A}$ and $\mathcal{B}$. Step 1 runs in constant time. If we use Floyd–Warshall algorithm for finding the diameter of a graph, then the complexity of step 2 is $O(n^3)$. Step 3 executes in $O(n)$ time. After the execution of Floyd–Warshall algorithm, all pairs of vertices between which the distance

is equal to $diam(G)$ will be known. Further, step 4 runs in $O(n^2)$ time. Based on these facts, the total execution time of the algorithm is $O(n^3)$. □

Theorem 5.9. *A tree G has a nontrivial convex 2-partition if and only if one of the following conditions holds*:

a) $diam(G) \geq 5$;

b) $G \in \mathcal{A}$;

c) $G \in \mathcal{B}$.

Proof. It is clear that if a tree G has a nontrivial convex 2-partition, then inequality $n \geq 6$ holds. Let us analyze nontrivial convex 2-partitions of G in dependence on its diameter.

Suppose $diam(G) = 2$. Here G is a star graph. It can simply be verified that a star graph has no nontrivial convex 2-partition.

Suppose $diam(G) = 3$. We choose two vertices $x, x' \in X(G)$ for which there is a path $L = [x, y, z, x']$ of length 3. Evidently, L is a unique path between vertices x and x', and vertices x, x' are terminal, i.e., $\Gamma(x) = y$ and $\Gamma(x') = z$. It follows from relation $|X(G)| \geq 6$ that G contains at least two vertices different from x, y, z, x'. Assume that there is a vertex $v \in X(G)$ different from x, y, z, x', and $v \in R_L(y)$ such that $d(y, v) \geq 2$, or $v \in R_L(z)$ and $d(z, v) \geq 2$. We get a contradiction because $d(y, x') = d(z, x) = 2$ and the length of the paths $L^1 = [x', z, y, \dots, v]$ and $L^2 = [x, y, z, \dots, v]$ is greater than or equal to 4. Consequently, all vertices of G different from x, y, z and x' are adjacent only to y or z. It can easily be checked that if y is adjacent only to x and z, or z is adjacent only to x' and y, then G has no nontrivial convex 2-partition. In the converse case, G has a nontrivial convex 2-partition

$$\mathcal{P}(G) = \{\{x, y\} \cup R_L(y), \{z, x'\} \cup R_L(z)\}.$$

In other words, if $diam(G) = 3$, then G has a nontrivial convex 2-partition if and only if $G \in \mathcal{A}$.

Suppose $diam(G) = 4$. We choose two vertices $x, x' \in X(G)$ such that there is a path $L = [x, y, z, h, x']$. The length of L is equal to the diameter of G and vertices x and x' are terminal in G. Since $|X(G)| \geq 6$, G contains at least one vertex v different from x, y, z, h, x'. If v is adjacent to y or h, then G has a nontrivial convex 2-partition

$$\mathcal{P}(G) = \{\{x, y\} \cup R_L(y), \{z, h, x'\} \cup R_L(z) \cup R_L(h)\} \text{ or}$$

$$\mathcal{P}(G) = \{\{x, y, z\} \cup R_L(y) \cup R_L(z), \{h, x'\} \cup R_L(h)\}, \text{ respectively.}$$

Assume that there are no vertices different from x, y, z, h, x', which are adjacent to y or h. Then, there exist vertices z' different from y and h which are adjacent to z. If $|\Gamma(z')| = 1$ or $|\Gamma(z')| = 2$ for all such z', then it is not hard to check that G has no nontrivial convex 2-partition. Now assume that there are at least two vertices z'' and z''' different from z and adjacent to z', i.e., $|\Gamma(z')| \geq 3$. In this case, we obtain a path $L = [z'', z', z, y, x]$. As mentioned above, it follows that G has a nontrivial convex 2-partition. Equivalently, if $diam(G) = 4$, then G has a nontrivial convex 2-partition if and only if $G \in \mathcal{B}$.

Suppose $diam(G) \geq 5$. We choose two vertices x and x' from $X(G)$ such that $d(x, x') = diam(G)$. Let $L = [x, x_1, x_2, \dots, x_k, x']$, $k \geq 4$, be a path between x and x'. L contains at least 6 vertices. Moreover, L is a unique path between x and x'. Hence, paths $[x, x_1, x_2]$ and $[x_3, \dots, x_k, x']$ generate a nontrivial convex 2-partition of G as follows:

$$\mathcal{P}(G) = \left\{\{x\} \cup \bigcup_{i=1}^{2} R_L(x_i), \{x'\} \cup \bigcup_{i=3}^{k} R_L(x_i)\right\}.$$

The theorem is proved. □

Theorem 5.10. *If a tree G on $n \geq 6$ vertices has a nontrivial convex partition, then G has a nontrivial convex p-partition for every p, $2 \leq p \leq \theta_{cn}^{max}(G)$.*

Proof. If a tree G has a nontrivial convex partition, then there is a maximum nontrivial convex partition $\mathcal{P}_{\theta_{cn}^{max}}(G)$. If $\theta_{cn}^{max}(G) = 2$, then the theorem is proved. If $\theta_{cn}^{max}(G) \geq 3$, then after repeating $\theta_{cn}^{max}(G) - 2$ times the procedure described in the proof of Theorem 5.3, we obtain a nontrivial convex 2-partition of G. It follows that G has a nontrivial convex p-partition, for every p, $2 \leq p \leq \theta_{cn}^{max}(G)$. □

The following corollaries are true.

Corollary 5.5. *If a tree G on $n \geq 6$ vertices has a nontrivial convex partition, then G has a nontrivial convex 2-partition.*

Corollary 5.6. *A tree G has a nontrivial convex p-partition for every p, $2 \leq p \leq \theta_c^{max}(G)$, if and only if one of the following conditions holds*:

a) $diam(G) \geq 5$*;*

b) $G \in \mathcal{A}$*;*

c) $G \in \mathcal{B}$*.*

Let x be a vertex of G for which $|\Gamma(x) \cap C(G)| \geq 2$ or there is another vertex $y \in \Gamma(x)$ such that $\Gamma(y) = \{x, z\}$, $z \in C(G)$, where $C(G)$ is the set of all terminal vertices of G.

For x that satisfies the announced properties, we define the following set:

$$S_x = \{x\} \cup \{v \in X(G) : v \in \Gamma(x) \cap C(G)\} \cup$$
$$\{y, z \in X(G) : \Gamma(y) = \{x, z\}, z \in C(G)\}.$$

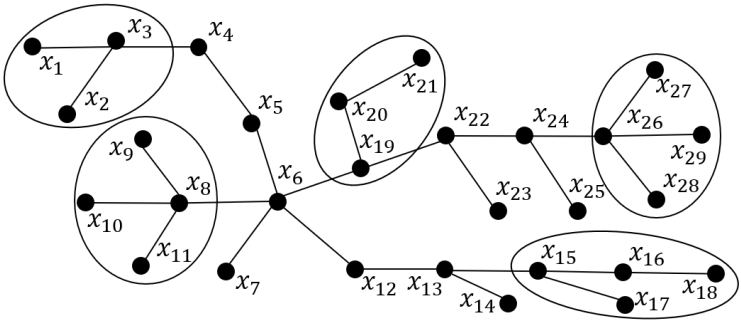


Figure 5.3. *The family $\mathcal{S}(G)$ of a tree G.*

The set S_x is called the *nontrivial terminal set* of G. Note that S_x is a nontrivial convex set of G. We say that a terminal vertex z of a tree G *corresponds* to a nontrivial terminal set S_x of G if S_x contains z.

By $\mathcal{S}(G)$ we denote the family of all nontrivial terminal sets of a tree G. For example, for G presented in Figure 5.3 we have the following family of nontrivial terminal sets:

$$\mathcal{S}(G) = \{S_{x_3} = \{x_1, x_2, x_3\}, S_{x_8} = \{x_8, x_9, x_{10}, x_{11}\},$$
$$S_{x_{15}} = \{x_{15}, x_{16}, x_{17}, x_{18}\}, S_{x_{19}} = \{x_{19}, x_{20}, x_{21}\},$$
$$S_{x_{26}} = \{x_{26}, x_{27}, x_{28}, x_{29}\}\}.$$

Lemma 5.2. *Let G be a tree. All nontrivial terminal sets of $\mathcal{S}(G)$ are disjoint.*

Proof. Suppose that there are at least two different nontrivial terminal sets S_x and S_y of $\mathcal{S}(G)$ such that $S_x \cap S_y \neq \emptyset$. In this case, by the definition of nontrivial terminal set, we have $S_x = S_y$. Whence, we obtain a contradiction. □

Lemma 5.3. *$\mathcal{S}(G)$ is unique for a tree G.*

Proof. Correctness of the lemma results from the definition of nontrivial terminal set and Lemma 5.2. □

Lemma 5.4. *For a tree G, every set of $\mathcal{S}(G)$ belongs to exactly one set of a maximum nontrivial convex cover $\mathcal{P}_{\theta_{cn}^{max}}(G)$ such that any two nontrivial terminal sets of $\mathcal{S}(G)$ do no belong to the same set of $\mathcal{P}_{\theta_{cn}^{max}}(G)$.*

Proof. From the definitions of nontrivial terminal set and nontrivial convex partition, it follows that every set of $\mathcal{S}(G)$ belongs to exactly one set of $\mathcal{P}_{\theta_{cn}^{max}}(G)$. Suppose that there is a set C of $\mathcal{P}_{\theta_{cn}^{max}}(G)$ that contains at least two different nontrivial terminal sets of G. Let $\mathcal{S}_C$ be the family of all nontrivial terminal sets which are in C and $k = |\mathcal{S}_C| \geq 2$. By Lemmas 5.2 and 5.3, we know that $\mathcal{S}(G)$ is unique for G and all nontrivial terminal sets are disjoint. Further, we separate C into disjoint nontrivial convex sets $S_1, S_1, \ldots, S_k$, where every set contains exactly one nontrivial terminal set of $\mathcal{S}_C$. If there are some vertices of C which remain uncovered by new nontrivial convex sets $S_1, S_1, \ldots, S_k$, then we select one of such vertices, say x, that is adjacent to a vertex $y \in S$, $S \in \{S_1, S_1, \ldots, S_k\}$, and further add x to S. If some uncovered vertices still remain, then we repeat the above procedure. Since $k \geq 2$, we get a new convex cover $\mathcal{P}(G)$ such that $|\mathcal{P}(G)| > |\mathcal{P}_{\theta_{cn}^{max}}(G)|$. Hence, we have a contradiction. □

Lemma 5.5. *A tree G with $2 \leq diam(G) \leq 4$ has at least one nontrivial terminal set.*

Proof. From the definition of nontrivial terminal set, it follows that every tree G with $diam(G) = 2$ contains exactly one nontrivial terminal set $S_x = X(G)$. It can easily be checked that a tree $G \in \mathcal{A}$ has exactly two nontrivial terminal sets, and any tree $G \in \mathcal{B}$ has at least two nontrivial

terminal sets. Similarly, if a tree G with $diam(G) = 3$ does not belong to the class $\mathcal{A}$, or $diam(G) = 4$ and $G \notin \mathcal{B}$, then G has exactly one nontrivial terminal set $S_x = X(G)$. □

Lemma 5.6. *A tree G with $diam(G) \geq 5$ has at least two nontrivial terminal sets.*

Proof. Let G be a tree with $diam(G) \geq 5$. Let x and y be two terminal vertices such that $d(x, y) = diam(G)$. Assume that x does not correspond to any nontrivial terminal set. By the definition of nontrivial terminal set, x is adjacent to a vertex z that is adjacent to at least two vertices different from x and all of them are nonterminal. Let $z_1, z_2, \dots, z_k$, where $k \geq 2$, be vertices different from x and adjacent to z. The path between vertices x and y contains exactly one vertex $z' \in \{z_1, z_2, \dots, z_k\}$. Since $z_1, z_2, \dots, z_k$ are nonterminal vertices, for every vertex $z'' \in \{z_1, z_2, \dots, z_k\} \backslash \{z'\}$ there exists a vertex z^* different from z and adjacent to z''. Since for any two vertices of G there is only one path connecting them, for every vertex z^*, we get $d(z^*, y) > diam(G)$. Consequently, we obtain a contradiction. Similarly, we get a contradiction if we assume that y does not correspond to any nontrivial terminal set. Taking into account that $diam(G) \geq 5$, vertices x and y correspond to different nontrivial terminal sets. Hence, a connected tree G with $diam(G) \geq 5$ has at least two nontrivial terminal sets. □

Algorithm 5.2. *Determines $\mathcal{S}(G)$ for a tree G.*

Input: *A tree G.*

Output: $\mathcal{S}(G)$.

Step 1. *Define the set $\mathcal{S}(G) = \emptyset$.*

Step 2. *Determine all terminal vertices $C(G)$.*

Step 3. *Go through all vertices $x \in X(G) \backslash C(G)$. If for a vertex x of G we have $|\Gamma(x) \cap C(G)| \geq 2$ or there is another vertex $y \in \Gamma(x)$ such that $\Gamma(y) = \{x, z\}$, where $z \in C(G)$, then we define the set of vertices $S_x = \{x\} \cup \{v \in X(G) : v \in \Gamma(x) \cap C(G)\} \cup \{y, z \in X(G) : \Gamma(y) = \{x, z\}, z \in C(G)\}$ and add it to $\mathcal{S}(G)$.*

Step 4. *Return $\mathcal{S}(G)$.*

Theorem 5.11. *Algorithm* 5.2 *determines the family of nontrivial terminal sets* $\boldsymbol{\mathcal{S}}(G)$ *of a tree* G *in* $O(n^2)$ *time.*

Proof. Correctness of the algorithm results from Lemmas 5.2 – 5.6. Clearly, steps 1 and 4 run in constant time. Step 2 operates in $O(n)$, and step 3 executes in $O(n^2)$ time. So, the total execution time of the algorithm is $O(n^2)$. □

For a tree G, by $\boldsymbol{\mathcal{E}}(G)$ we denote the family of all subtrees obtained after the elimination of all nontrivial terminal sets of $\boldsymbol{\mathcal{S}}(G)$ from G.

For the sake of estimation of the number $\theta_{cn}^{max}(G)$, we consider that if $0 \leq |X(G)| \leq 2$, then $\theta_{cn}^{max}(G) = 0$. Likewise, if $|X(G)| = 3$, then $\theta_{cn}^{max}(G) = 1$. The next theorem establishes the recursive formula for determining $\theta_{cn}^{max}(G)$.

Theorem 5.12. *For a tree* G*, the following relation holds*:

$$\theta_{cn}^{max}(G) = \begin{cases} |\boldsymbol{\mathcal{S}}(G)| + \sum\limits_{G' \in \boldsymbol{\mathcal{E}}(G)} \theta_{cn}^{max}(G'), & if\ |X(G)| \geq 3; \\ 0, & if\ 0 \leq |X(G)| \leq 2. \end{cases}$$

Proof. By Lemma 5.4, we conclude that by eliminating all nontrivial terminal sets of $\boldsymbol{\mathcal{S}}(G)$ from tree G, we eliminate minimal nontrivial convex sets of G containing nontrivial terminal sets. Besides, after the elimination of all nontrivial terminal sets of family $\boldsymbol{\mathcal{S}}(G)$ from G, we obtain a family of subtrees $\boldsymbol{\mathcal{E}}(G)$ such that some of them also contain nontrivial terminal sets.

If $0 \leq |X(G)| \leq 2$, then $\theta_{cn}^{max}(G) = 0$. On the contrary case, if $|X(G)| \geq 3$, then taking into account Lemmas 5.2 – 5.6, we get the following relation:

$$\theta_{cn}^{max}(G) = |\boldsymbol{\mathcal{S}}(G)| + \sum_{G' \in \boldsymbol{\mathcal{E}}(G)} \theta_{cn}^{max}(G').$$

The theorem is proved. □

Next, we propose a recursive procedure $Max\theta(G)$ that determines the maximum nontrivial convex partition number $\theta_{cn}^{max}(G)$ of a tree G. After, we prove that this procedure executes in polynomial time.

Procedure $\boldsymbol{Max\theta(G)}$. *Determines $\theta_{cn}^{max}(G)$ for a tree G.*

Input: *A tree G.*

Output: $\theta_{cn}^{max}(G)$.

Step 1. If $0 \leq |X(G)| \leq 2$, then return 0.

Step 2. Apply Algorithm 5.2, i.e., determine $\boldsymbol{\mathcal{S}}(G)$. Remove every nontrivial terminal set of $\boldsymbol{\mathcal{S}}(G)$ from G and obtain $\boldsymbol{\mathcal{E}}(G)$.

Step 3. For every subtree G' of $\boldsymbol{\mathcal{E}}(G)$, apply procedure $Max\theta(G')$ and, after all, return $\theta_{cn}^{max}(G) = |\boldsymbol{\mathcal{S}}(G)| + \sum_{G' \in \boldsymbol{\mathcal{E}}(G)} Max\theta(G')$.

Theorem 5.13. *Procedure $Max\theta(G)$ determines the number $\theta_{cn}^{max}(G)$ of a tree G in $O(n^3)$ time.*

Proof. From Theorem 5.12, we know that for a tree G, the procedure $Max\theta(G)$ returns the number $\theta_{cn}^{max}(G)$. In the general case, by Lemma 5.6 and Theorem 5.11, the processing time of procedure $Max\theta(G)$ is

$$T(n) = \sum_{i=1}^{k} T(n_i) + O(n^2),$$

where $\sum_{i=1}^{k} n_i \leq n - 6$, $k \geq 1$, $|\boldsymbol{\mathcal{E}}(G)| = k$, and n_i is the number of vertices of the subtree G_i of $\boldsymbol{\mathcal{E}}(G)$.

The worst-case behavior of procedure $Max\theta(G)$ occurs when every tree (except the trivial case of a tree whose vertices form a nontrivial terminal set), examined by the procedure, has exactly two nontrivial terminal sets, which consist of three elements such that after their elimination a single subtree remains. In this case, if $n \geq 6$, the processing time of $Max\theta(G)$ is

$$T(n) = T(n-6) + O(n^2).$$

To compute $T(n)$, we use the following recurrence relation:

$$T(n) = T(n-6) + cn^2,$$

where $c > 0$ is a constant. Let us assume that $n = k * 6$, for some natural number k. Expanding the recurrence, we get

$$\begin{aligned} T(n) &= T(n-6) + cn^2 = \\ &= T(n-12) + c(n-6)^2 + cn^2 = \end{aligned}$$

$$= \cdots$$

$$= \sum_{i=0}^{n/6} c(n-6i)^2 =$$

$$= c\left(n^2 \sum_{i=0}^{n/6} 1 + 36 \sum_{i=0}^{n/6} i^2 - 12n \sum_{i=0}^{n/6} i \right).$$

Applying the equalities

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

for the three sums above, we get:

$$n^2 \sum_{i=0}^{n/6} 1 = n^2 \left(\frac{n+6}{6}\right) = \frac{1}{6}(n^3 + 6n^2),$$

$$36 \sum_{i=0}^{n/6} i^2 = \frac{36}{6}\left(\frac{n}{6}\left(\frac{n+6}{6}\right)\left(\frac{2n+6}{6}\right)\right) = \frac{1}{18}(n^3 + 9n^2 + 18n),$$

$$12n \sum_{i=0}^{n/6} i = \frac{12n}{2}\left(\frac{n}{6}\left(\frac{n+6}{6}\right)\right) = \frac{1}{6}(n^3 + 6n^2).$$

Now we substitute back in these three simplifications:

$$T(n) = c\left(\frac{1}{6}(n^3 + 6n^2) + \frac{1}{18}(n^3 + 9n^2 + 18n) - \frac{1}{6}(n^3 + 6n^2)\right) =$$

$$= \frac{c}{18}(n^3 + 9n^2 + 18n).$$

From the above, it is clear that

$$T(n) = O(n^3).$$

Therefore, procedure $Max\theta(G)$ determines the number $\theta_{cn}^{max}(G)$ in $O(n^3)$ time. The theorem is proved. □

Corollary 5.7. *It can be decided in $O(n^3)$ time whether a tree G on $n \geq 6$ vertices has a nontrivial convex p-partition for every p, $2 \leq p \leq \left\lfloor \frac{n}{3} \right\rfloor$.*

We illustrate an example of the determination of the number $\theta_{cn}^{max}(G)$ for a tree G from Figure 5.3.

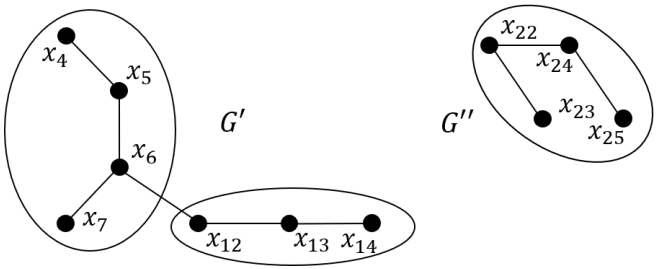


Figure 5.4. *The family of subtrees $\mathcal{E}(G) = \{G', G''\}$ obtained after removing from G (Figure 5.3) the sets belonging to the family $\mathcal{S}(G)$.*

The family of nontrivial terminal sets of G is

$$\mathcal{S}(G) = \{S_{x_3} = \{x_1, x_2, x_3\}, S_{x_8} = \{x_8, x_9, x_{10}, x_{11}\},$$
$$S_{x_{15}} = \{x_{15}, x_{16}, x_{17}, x_{18}\},\ S_{x_{19}} = \{x_{19}, x_{20}, x_{21}\},$$
$$S_{x_{26}} = \{x_{26}, x_{27}, x_{28}, x_{29}\}\}.$$

After removing from G the sets belonging to the family $\mathcal{S}(G)$, we obtain the family of subtrees $\mathcal{E}(G) = \{G', G''\}$, represented in Figure 5.4, to which correspond the following families of nontrivial terminal sets:

$$\mathcal{S}(G') = \{S_{x_6} = \{x_4, x_5, x_6, x_7\}, S_{x_{12}} = \{x_{12}, x_{13}, x_{14}\}\},$$
$$\mathcal{S}(G'') = \{S_{x_{22}} = X(G'') = \{x_{22}, x_{23}, x_{24}, x_{25}\}\}.$$

Now, after removing from G' the sets belonging to the family $\mathcal{S}(G')$, and from G'' the set belonging to the family $\mathcal{S}(G'')$, the null graphs are obtained.

So, we have $\theta_{cn}^{max}(G') = |\mathcal{S}(G')| = 2$, $\theta_{cn}^{max}(G'') = |\mathcal{S}(G'')| = 1$ and $|\mathcal{S}(G)| = 5$. Therefore, we get the maximum nontrivial convex partition number of G:

$$\theta_{cn}^{max}(G) = |\mathcal{S}(G)| + \theta_{cn}^{max}(G') + \theta_{cn}^{max}(G'') = 5 + 2 + 1 = 8.$$

Chapter 6
Graph Operations

This chapter studies minimum convex covers and maximum nontrivial convex covers of graphs resulting from some graph operations.

A nonempty subset S of $X(G)$ is *a non-connecting* set in a graph G if for every pair of vertices $x, y \in X(G)\backslash S$ with $d(x, y) = 2$, the following relation holds $\Gamma(x) \cap \Gamma(y) \cap S = \emptyset$.

For any set $S \subseteq X(G)$ and $y \in S$, we say that vertex $x \in X(G)\backslash S$ is an *external private neighbor* of y *with respect to* S if $\Gamma(x) \cap S = \{y\}$. A set $S \subseteq X(G)$ is said to be a *private dominating set* of a graph G if it is a dominating set, and for every $y \in S$ there exists an external private neighbor $x \in X(G)\backslash S$.

A minimum number of cliques covering all vertices of a graph G is known as a *clique cover number* $\theta(G)$. The *minimum clique cover* $\mathcal{P}_\theta(G)$ of G is the clique cover such that $|\mathcal{P}_\theta(G)| = \theta(G)$.

By $P(G)$ we denote a family of convex sets such that $X(G) = \bigcup_{S \in P(G)} S$. We denote by $\mathcal{P}(P(G))$ a convex cover of G that consists of sets, which belong to $P(G)$. Note that for a given $P(G)$, which does not have the set $X(G)$, we can obtain a convex cover $\mathcal{P}(P(G))$ by removing from $P(G)$ all sets contained in the union of other sets of $P(G)$.

The following propositions are true:

Proposition 6.1. *Let G be a connected graph of order $n \geq 2$. Then for every $x \in X(G)$ there is a convex set $S \subseteq X(G)$ such that $x \in S$, $|S| = 2$.*

Proposition 6.2. *Let G be a connected graph of order $n \geq 3$. There exists $\mathcal{P}_{\varphi_c^{min}}(G)$ such that for every set $S \in \mathcal{P}_{\varphi_c^{min}}(G)$, $|S| \geq 2$.*

Proposition 6.3. *Let G be a connected graph of order $n \geq 3$. There exists $\mathcal{P}_\theta(G)$ such that for every set $S \in \mathcal{P}_\theta(G)$, $|S| \geq 2$.*

Theorem 6.1. *Let G be a connected graph of order $n \geq 3$ that contains a universal vertex. Then for every vertex $g \in X(G)$ there is a convex set $S \subseteq X(G)$ such that $g \in S$ and $|S| = 3$.*

Proof. Let x be a universal vertex of G, i.e., $\Gamma(x) = X(G)\backslash\{x\}$. Suppose that $G[\Gamma(x)]$ is a disconnected graph. Then, for any two vertices $x_1 \in X(G_1)$ and $x_2 \in X(G_2)$, where G_1 and G_2 are two different connected components of $G[\Gamma(x)]$, we obtain a convex set $S = \{x, x_1, x_2\}$, $|S| = 3$.

Now suppose that $G[\Gamma(x)]$ is a connected graph. In this case, every vertex y of $X(G)\backslash\{x\}$ has an adjacent vertex $z \in X(G)\backslash\{x\}$. Therefore, $S = \{x, y, z\}$ is a convex set and $|S| = 3$. □

Corollary 6.1. *Let G be a connected graph of order $n \geq 4$ that contains a universal vertex. Then G has a nontrivial convex cover.*

Corollary 6.2. *Let G be a connected graph of order $n \geq 4$ that contains a universal vertex. Then $\varphi_c^{min}(G) = \varphi_{cn}^{min}(G)$.*

6.1 Join

The *join* of two graphs G and H, denoted $G + H$, is the graph with vertex set $X(G + H) = X(G) \cup X(H)$ and edge set:

$$U(G + H) = U(G) \cup U(H) \cup \{xy : x \in X(G), y \in X(H)\}.$$

Theorem 6.2. [21] *Let G be a connected graph and K_m a complete graph of order m. Then a proper subset $C = S_1 \cup S_2$ of $X(G + K_m)$, where $S_1 \subseteq X(G)$ and $S_2 \subseteq X(K_m)$, is convex in $G + K_m$ if and only if either*

a) S_1 is a clique in G, or

b) $S_1 \subseteq X(G)\backslash S$ and $S_2 = X(K_m)$ for some non-connecting set S of graph G.

Theorem 6.3. *Let G be a connected graph with $diam(G) = 2$ and K_m a complete graph of order $m \geq 1$. Let $C = S_1 \cup S_2$ be a proper convex subset of $X(G + K_m)$, where $S_1 \subseteq X(G)$ and $S_2 \subseteq X(K_m)$. Then S_1 is convex in G.*

Proof. By Theorem 6.2, let us consider two cases. Firstly, if S_1 is a clique in G, then evidently it is convex in G. It can be assumed that S_1 is not a clique in G. Thus, $S_2 = X(K_m)$ and $S_1 \subseteq X(G)\backslash S$ for some non-connecting set S of G. Assume further that S_1 is not convex in G. Let x and y be two vertices of S_1 such that there exists a vertex $z \in \langle x, y\rangle_G$ that does not belong to S_1. Since the diameter of G is 2, we have $d_G(x, y) = 2$ and $z \in \Gamma_G(x) \cap \Gamma_G(y)$. Hence, $z \in S_1$. Therefore, there is a contradiction. So, S_1 is convex in G. □

Theorem 6.4. *Let G be a connected graph of order $n \geq 1$ and K_m a complete graph of order $m \geq 1$. Then the following statements hold:*

1) $\varphi_c^{min}(G + K_m) = 2$ if G is complete;

2) $\varphi_{cn}^{min}(G + K_m) = 2$ if G is complete and $n + m \geq 4$;

3) $\varphi_c^{min}(G + K_m) = \varphi_{cn}^{min}(G + K_m) = \varphi_c^{min}(G)$ if $diam(G) = 2$;

4) $\varphi_c^{min}(G + K_m) = \varphi_{cn}^{min}(G + K_m) \leq \varphi_c^{min}(G)$ if $diam(G) \geq 3$.

Proof. 1) Suppose $G = K_n$. Then, by the definition of join of two graphs, it follows that $G + K_m$ also is complete. Here both graphs K_n and K_m have at least one vertex. Further, we obtain

$$\varphi_c^{min}(G + K_m) = 2.$$

2) Suppose $G = K_n$ and $n + m \geq 4$. As before, graph $G + K_m$ is complete, and we get

$$\varphi_{cn}^{min}(G + K_m) = 2.$$

3) Suppose G is a non-complete graph with $diam(G) = 2$. Let C be a proper convex subset of $X(G + K_m)$ that satisfies the conditions of Theorem 6.2. It follows from Theorem 6.3 that $X(G) \cap C$ is a convex set in G. Let $\boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G + K_m)$ be a minimum convex cover of $G + K_m$. We get a family $P(G) = \bigcup_{S \in \boldsymbol{\mathcal{P}}_{\varphi_c^{min}(G+K_m)}} \{X(G) \cap S\}$. It is clear that $P(G)$ does not contain the set $X(G)$. Hence, $|\boldsymbol{\mathcal{P}}(P(G))| \leq \varphi_c^{min}(G + K_m)$. We obtain

$$\varphi_c^{min}(G) \leq \varphi_c^{min}(G + K_m).$$

By Proposition 6.2, a connected graph G on $n \geq 3$ vertices has a minimum convex cover $\boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G)$ such that for every set $S \in \boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G)$,

the condition $|S| \geq 2$ holds. Hence, we obtain a nontrivial convex cover $\mathcal{P}(G + K_m)$ of $G + K_m$ by adding $X(K_m)$ to Y_i, where $Y_i \in \mathcal{P}_{\varphi_c^{min}}(G)$, for each i, $1 \leq i \leq \varphi_c^{min}(G)$. Note that $|\mathcal{P}(G + K_m)| = \varphi_c^{min}(G)$ and the next inequalities hold:

$$\varphi_c^{min}(G + K_m) \leq \varphi_{cn}^{min}(G + K_m) \leq \varphi_c^{min}(G)$$

As a consequence, we obtain

$$\varphi_c^{min}(G + K_m) = \varphi_{cn}^{min}(G + K_m) = \varphi_c^{min}(G).$$

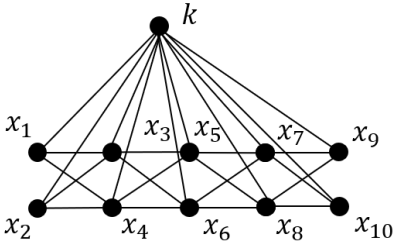


Figure 6.1. *The graph* $H + K_1$, $X(K_1) = \{k\}$, $\varphi_{cn}^{min}(H + K_1) < \varphi_c^{min}(H)$.

4) Now, assume that G is a non-complete graph and $diam(G) \geq 3$. As above, it is easy to prove that every minimum convex cover of G, which satisfies Proposition 6.2, generates a nontrivial convex cover of $G + K_m$. Whence, we get

$$\varphi_{cn}^{min}(G + K_m) \leq \varphi_c^{min}(G).$$

Note also that there exist non-complete graphs H, $diam(H) \geq 3$, for which the inequality $\varphi_{cn}^{min}(H + K_m) < \varphi_c^{min}(H)$ holds. For instance, the graph represented in Figure 6.1 is the join of graphs H and K_1, where $X(K_1) = \{k\}$. This graph has a minimum nontrivial convex cover

$$\mathcal{P}_{\varphi_{cn}^{min}}(H + K_1) = \big\{\{x_1, x_7, x_9, k\}, \{x_2, x_8, x_{10}, k\}, \{x_3, x_5, k\}, \{x_4, x_6, k\}\big\},$$

and this convex cover consists of 4 nontrivial sets, but the graph H has a minimum convex cover

$$\mathcal{P}_{\varphi_c^{min}}(W) = \big\{\{x_1, x_3\}, \{x_5, x_7\}, \{x_2, x_4\}, \{x_6, x_8\}, \{x_9\}, \{x_{10}\}\big\}.$$

It follows that for this graph, we have $\varphi_{cn}^{min}(H + K_1) = 4$, $\varphi_c^{min}(H) = 6$ and $\varphi_{cn}^{min}(H + K_1) < \varphi_c^{min}(H)$.

We stress that a nontrivial convex cover is a particular case of a convex cover. Since any vertex of $X(K_m)$ is universal in $G + K_m$, Corollary 6.2 implies that $\varphi_c^{min}(G + K_m) = \varphi_{cn}^{min}(G + K_m)$.

Therefore, we get

$$\varphi_c^{min}(G + K_m) = \varphi_{cn}^{min}(G + K_m) \leq \varphi_c^{min}(G).$$

The theorem is proved. □

Before proceeding further with analysis of the maximum nontrivial convex cover number of graphs resulting from the join of two graphs, we state the following important result, which guarantees that the minimum cardinality of a private dominating set in a graph G without isolated vertices always equals $\gamma(G)$.

Theorem 6.5. [22] *Every graph G without isolated vertices has a minimum dominating set which is also a private dominating set.*

Theorem 6.6. *Let G be a connected graph of order n and K_m a complete graph of order m. Then the following statements hold:*

1) $\varphi_{cn}^{max}(G + K_m) \geq n - \gamma(G)$ *if* $n \geq 3$ *and* $m = 1$*;*

2) $\varphi_{cn}^{max}(G + K_m) = n + m - 2$ *if* $n = 1$ *and* $m \geq 3$*, or* $n \geq 2$ *and* $m \geq 2$.

Proof. 1) Assume that $n \geq 3$ and $m = 1$. We define a family of nontrivial convex sets $\mathcal{P}(G + K_1) = \emptyset$ that will cover graph $G + K_1$. By Theorem 6.5, there exists a private dominating set D of G with cardinality $\gamma(G)$. We add the set of vertices $\{x, y, z\}$ to $\mathcal{P}(G + K_1)$ for each $x \in X(G)\backslash D$, where $y \in \Gamma(x) \cap D$ and $z \in X(K_1)$. Consequently, $|\mathcal{P}(G + K_1)| = n - \gamma(G)$ and thus

$$\varphi_{cn}^{max}(G + K_1) \geq n - \gamma(G).$$

2) Assume that $n = 1$ and $m \geq 3$. We define a family of nontrivial and convex sets $\mathcal{P}(G + K_m) = \emptyset$ that will cover $G + K_m$, and choose two vertices $x, y \in X(K_m)$, $x \neq y$. Then, for each $z \in X(G + K_m)\backslash\{x, y\}$, we add the set $\{x, y, z\}$ to $\mathcal{P}(G + K_m)$. Since every set of $\mathcal{P}(G + K_m)$ consists of three vertices, two of which are x and y and the third vertex is resident in $\mathcal{P}(G + K_m)$, it follows that $\mathcal{P}(G + K_m)$ is a maximum nontrivial convex cover of $G + K_m$ and

$$\varphi_{cn}^{max}(G + K_m) = n + m - 2.$$

3) If $n \geq 2$ and $m \geq 2$, then by following the same arguments as described above, we get

$$\varphi_{cn}^{max}(G + K_m) = n + m - 2.$$

The theorem is proved. □

Theorem 6.7. [21] *Let G and H be connected non-complete graphs. Then a proper subset $C = S_1 \cup S_2$ of $X(G + H)$, where $S_1 \subset X(G)$, $S_2 \subset X(H)$, is convex in $G + H$ if and only if S_1 and S_2 are cliques in G and H respectively.*

Theorem 6.8. *Let G and H be two connected non-complete graphs. Then:*

$$\theta(G + H) = \varphi_c^{min}(G + H) = \varphi_{cn}^{min}(G + H) = max\{\theta(G), \theta(H)\}.$$

Proof. From Theorem 6.7, we know that every convex set of $G + H$ is a clique. Further, every convex cover of $G + H$ is a clique cover and $\varphi_c^{min}(G + H) = \theta(G + H)$. Let $\boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G + H)$ be a minimum convex cover of $G + H$. By Theorem 6.7, we obtain a family of sets

$$P(G) = \bigcup_{S \in \boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G+H)} \{X(G) \cap S\}$$

that has no set $X(G)$, and every set of $P(G)$ is a clique. This implies the inequality $|\boldsymbol{\mathcal{P}}(P(G))| \leq \varphi_c^{min}(G + H)$. Thus, $\theta(G) \leq \varphi_c^{min}(G + H)$. In the same way, we get $|\boldsymbol{\mathcal{P}}(P(H))| \leq \varphi_c^{min}(G + H)$, for

$$P(H) = \bigcup_{S \in \boldsymbol{\mathcal{P}}_{\varphi_c^{min}}(G+H)} \{X(H) \cap S\}.$$

Further, we get $\theta(H) \leq \varphi_c^{min}(G + H)$, and hence

$$max\{\theta(G), \theta(H)\} \leq \varphi_c^{min}(G + H).$$

By Proposition 6.3, we consider the minimum clique covers $\boldsymbol{\mathcal{P}}_\theta(G)$ and $\boldsymbol{\mathcal{P}}_\theta(H)$ of G and H, respectively, such that every set of $\boldsymbol{\mathcal{P}}_\theta(G)$ and $\boldsymbol{\mathcal{P}}_\theta(H)$ has at least two vertices. If $\theta(G) \geq \theta(H)$, then further we construct a nontrivial clique cover $\boldsymbol{\mathcal{P}}(G + H)$, $|\boldsymbol{\mathcal{P}}(G + H)| = \theta(G)$. Since every convex set of $G + H$ is a clique, we combine the sets X_i and Y_i together, where $X_i \in \boldsymbol{\mathcal{P}}_\theta(G)$ and $Y_i \in \boldsymbol{\mathcal{P}}_\theta(H)$, for each $i, 1 \leq i \leq \theta(H)$, and after, we combine X_i and Y_1, for each i, $\theta(H) + 1 \leq i \leq \theta(G)$. By the analogy,

if $\theta(G) < \theta(H)$, then it can be constructed a nontrivial clique cover $\mathcal{P}(G+H)$, $|\mathcal{P}(G+H)| = \theta(H)$. Consequently, we obtain

$$\varphi_c^{min}(G+H) \leq \varphi_{cn}^{min}(G+H) \leq max\{\theta(G), \theta(H)\}.$$

Finally, we have

$$\theta(G+H) = \varphi_c^{min}(G+H) = \varphi_{cn}^{min}(G+H) = max\{\theta(G), \theta(H)\}.$$

The theorem is proved. □

Theorem 6.9. *Let G and H be connected non-complete graphs of order n and m respectively. Then $\varphi_{cn}^{max}(G+H) \geq n+m-4$.*

Proof. We define a family of nontrivial convex sets $\mathcal{P}(G+H) = \emptyset$ that will cover $G+H$. Now, we choose four vertices $g_1, g_2 \in X(G)$ and $h_1, h_2 \in X(H)$ such that $g_1 \sim g_2$, $h_1 \sim h_2$. Then, for each vertex $h \in X(H)\backslash\{h_1, h_2\}$ and for each $g \in X(G)\backslash\{g_1, g_2\}$, we add sets $\{g_1, g_2, h\}$ and $\{h_1, h_2, g\}$ to $\mathcal{P}(G+H)$. It can be easily verified that $|\mathcal{P}(G+H)| = n+m-4$. So, $\varphi_{cn}^{max}(G+H) \geq n+m-4$. □

Theorem 6.10. *Let G and H be connected non-complete graphs of order n and m respectively. Then $\varphi_{cn}^{max}(G+H) = n+m-2$ if and only if there are two universal vertices in $G+H$.*

Proof. Let $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ be a maximum nontrivial convex cover of graph $G+H$ and $\varphi_{cn}^{max}(G+H) = n+m-2$. This entails that each set $S \in \mathcal{P}_{\varphi_{cn}^{max}}(G+K_m)$ has the cardinality three and contains exactly one resident vertex in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$. Moreover, there are two vertices x and y of $X(G+H)$, which are common for all sets of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$. By Theorem 6.7, it follows that $x \sim y$ and any vertex of $X(G+H)\backslash\{x, y\}$ is simultaneously adjacent to x and y. Therefore, vertices x and y are universal in $G+H$.

Assume that there are two universal vertices x and y in $G+H$, which in turn means that $x \sim y$. We define a family of nontrivial and convex sets $\mathcal{P}(G+H) = \emptyset$ that will cover $G+H$. We add the set of vertices $\{x, y, z\}$ to $\mathcal{P}(G+H)$ for each $z \in X(G+H)\backslash\{x, y\}$. It is clear that

$$\varphi_{cn}^{max}(G+H) = |\mathcal{P}(G+H)| = n+m-2.$$

The theorem is proved. □

Theorem 6.11. *Let G and H be connected non-complete graphs of order n and m respectively. Then $\varphi_{cn}^{max}(G+H) = n+m-3$ if and only if there are no two universal vertices in $G+H$, and there are three vertices x, y, z which induce a triangle or a path in $G+H$ such that for each vertex $v \in X(G)\backslash\{x,y,z\}$, there are two vertices $a, b \in \Gamma(v) \cap \{x,y,z\}$, $a \sim b$.*

Proof. Let $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ be a maximum nontrivial convex cover of graph $G+H$ such that $\varphi_{cn}^{max}(G+H) = n+m-3$. By considering Theorem 6.10, there are no two universal vertices in $G+H$. It follows from Theorem 6.7 that any convex set of $G+H$ is a clique. There is no set $S \in \mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ such that $|S| \geq 5$ because in this case we get $\varphi_{cn}^{max}(G+H) \leq n+m-4$. So, each set of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ contains no more than four vertices. We consider two cases.

1) Suppose there is a set $S \in \mathcal{P}_{\varphi_{cn}^{max}}(G+H)$, $|S| = 4$, containing two resident vertices r_1 and r_2 in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$. By replacing S with two sets $S_1 = S\backslash\{r_1\}$ and $S_2 = S\backslash\{r_2\}$ in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$, we obtain a new nontrivial convex cover of $G+H$ that has one more set than $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$. Have a contradiction. This implies that each set $S \in \mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ of cardinality 4 contains exactly one resident vertex in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ and, furthermore, all vertices of $X(G+H)\backslash S$ are resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ and belong to different sets of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$.

2) Suppose that every set of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ consists of three vertices. If there are two distinct sets S_1 and S_2 of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ for which $S_1 \cap S_1 = \emptyset$, then we obtain $\varphi_{cn}^{max}(G+H) \leq n+m-4$. This implies that the intersection of any two sets of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ is nonempty. Now assume that there are two sets of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$, each of which has two resident vertices, or there is only one set of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ containing two resident vertices and the number of non-resident vertices in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ is greater than or equal to 3. Combining this assumption with the fact that $|X(G+H)| \geq 6$, we again obtain $\varphi_{cn}^{max}(G+H) \leq n+m-4$. So, exactly one set of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ has two resident vertices and the number of non-resident vertices in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ is less than 3, or each set of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ contains exactly one resident vertex.

For both cases, let R be a set of resident vertices in $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$ that

contains exactly one vertex from each set of $\mathcal{P}_{\varphi_{cn}^{max}}(G+H)$. It is obvious that the set $\{x,y,z\} = \bigcup_{S\in\mathcal{P}_{\varphi_{cn}^{max}(G+H)}} S\backslash R$ induces a triangle or a path in $G+H$ such that for each vertex $v \in X(G)\backslash\{x,y,z\}$, there are two vertices $a,b \in \Gamma(v)\cap\{x,y,z\}$, $a \sim b$.

Assume that there are no two universal vertices in $G+H$ and there are three vertices x, y, z which induce a triangle or a path in $G+H$ such that for each vertex $v \in X(G)\backslash\{x,y,z\}$ there exist two different vertices $a,b \in \Gamma(v)\cap\{x,y,z\}$, $a \sim b$. From Theorems 6.9 and 6.10 follows:

$$n+m-4 \le \varphi_{cn}^{max}(G+H) \le n+m-3.$$

We define a family of nontrivial and convex sets $\mathcal{P}(G+H) = \emptyset$ that will cover $G+H$. Trivially, we add the set of vertices $\{v,a,b\}$ to $\mathcal{P}(G+H)$ for each $v \in X(H)\backslash\{x,y,z\}$, where $a,b \in \Gamma(v)\cap\{x,y,z\}$, $a \sim b$. Thus, $|\mathcal{P}(G+H)| = \varphi_{cn}^{max}(G+H) = n+m-3$. □

Summarizing Theorems 6.9 – 6.11, we get the following result.

Corollary 6.3. *Let G and H be connected non-complete graphs of order n and m respectively. Then $\varphi_{cn}^{max}(G+H) = n+m-4$ if and only if the following conditions hold:*

1) There are no two universal vertices in $G+H$;

2) There are no three vertices x, y, z, which induce a triangle or a path in $G+H$ such that for each vertex $v \in X(G)\backslash\{x,y,z\}$, there are two different vertices $a,b \in \Gamma(v)\cap\{x,y,z\}$, $a \sim b$.

6.2 Corona product

The *corona* of graphs G and H, denoted $G \odot H$, is the graph obtained by taking one copy of G and n copies of H, where $|X(G)| = n$, and then joining by an edge the ith vertex of graph G to every vertex in the ith copy of graph H.

We consider a general version of the corona of graphs. Let G be a connected graph with n vertices and let $\{g_1, g_2, \ldots, g_k\} \subseteq X(G)$, where $H_{g_1}, H_{g_2}, \ldots, H_{g_k}$, $1 \le k \le n$, be any graphs of order at least one. Then a graph obtained by taking one copy of graph G and after, by joining every vertex g_i to every vertex of the graph H_{g_i}, $1 \le i \le k$, is denoted by

$(G;\{g_1,g_2,\ldots,g_k\}) \odot \left(H_{g_1},H_{g_2},\ldots,H_{g_k}\right)$. If $H_{g_1}=\cdots=H_{g_k}=H$, then we simply denote $(G;\{g_1,g_2,\ldots,g_k\}) \odot H$. If also $k=n$, then $(G;\{g_1,g_2,\ldots,g_k\}) \odot H$ is the corona $G \odot H$.

Theorem 6.12. [23] *Let G be a connected graph and H be any graph with $\{g_1,g_2,\ldots,g_k\} \subseteq X(G)$ and $H_{g_1},H_{g_2},\ldots,H_{g_k}$ being the corresponding copies of H. A nonempty set $C \subseteq X(G \odot H)$ is convex in $G \odot H$ if and only if it satisfies one of the following conditions*:

a) C is a convex set in G;

b) C induces a complete subgraph of H_g for a vertex $g \in \{g_1,g_2,\ldots,g_k\}$;

c) $(G \odot H)[C] = (G[S];\{s_1,s_2,\ldots,s_l\}) \odot \left(H^*_{s_1},H^*_{s_2},\ldots,H^*_{s_l}\right)$, S is convex in G, $\{s_1,s_2,\ldots,s_l\} \subseteq S$, $\{s_1,s_2,\ldots,s_l\} \subseteq \{g_1,g_2,\ldots,g_k\}$, $X(s_i+H^*_{s_i})$ is convex in $s_i+H_{s_i}$ for each i, $1 \le i \le l$, where $H^*_{s_i}$ is a subgraph of H_{s_i}.

Theorem 6.13. *Let G and H be two connected graphs of order $n \ge 1$ and $m \ge 1$, respectively, with $\{g_1,g_2,\ldots,g_k\} \subseteq X(G)$, where $1 \le k \le n$. Then the following statements hold:*

1) $\varphi_c^{min}(G \odot H) = 2$ if $n = 1$ and H is complete;

2) $\varphi_{cn}^{min}(G \odot H) = 2$ if $n = 1$, H is complete and $m \ge 3$;

3) $\varphi_c^{min}(G \odot H) = \varphi_{cn}^{min}(G \odot H) = \varphi_c^{min}(H)$ if $n = 1$ and $diam(H) = 2$;

4) $\varphi_c^{min}(G \odot H) = \varphi_{cn}^{min}(G \odot H) \le \varphi_c^{min}(H)$ if $n = 1$ and $diam(H) \ge 3$;

5) $\varphi_c^{min}\left((G;\{g_1,g_2,\ldots,g_k\}) \odot H\right) = 2$ if $n \ge 2$;

6) $\varphi_{cn}^{min}\left((G;\{g_1,g_2,\ldots,g_k\}) \odot H\right) = 2$ if $n \ge 2$ and $k * m + n \ge 4$.

Proof. Suppose $n = 1$. In fact, we get

$$\varphi_c^{min}(K_1 \odot H) = \varphi_c^{min}(K_1 + H).$$

Consequently, statements 1), 2), 3), and 4) follow directly from Theorem 6.4.

5) Suppose $n \geq 2$. It can easily be checked that sets $X(H_{g_1})$ and $X(G) \cup \bigcup_{i=2}^{k} X(H_{g_i})$ satisfy the conditions of Theorem 6.12, and further these sets form a convex 2-cover of $(G;\{g_1, g_2, \ldots, g_k\}) \odot H$. This ensures the equality

$$\varphi_c^{min}\big((G;\{g_1, g_2, \ldots, g_k\}) \odot H\big) = 2.$$

6) Now suppose that $n \geq 2$ and $k * m + n \geq 4$. In other words, the set $X((G;\{g_1, g_2, \ldots, g_k\}) \odot H)$ contains at least 4. Taking into account Theorem 6.12, we have to analyze two cases:

a) If $m = 1$, then for a vertex $g \in \{g_1, g_2, \ldots, g_k\}$ we choose a vertex $g' \in \Gamma(g) \backslash X(H_g)$ that yields a nontrivial convex 2-cover

$$\mathcal{P}_2\big((G;\{g_1, g_2, \ldots, g_k\}) \odot H\big) = \Big\{\{g, g'\} \cup X(H_g),$$

$$X(G) \cup \bigcup\nolimits_{g'' \in \{g_1, g_2, \ldots, g_k\}, g'' \neq g} X\big(H_{g''}\big)\Big\}.$$

b) If $m \geq 2$, then for a vertex $g \in \{g_1, g_2, \ldots, g_k\}$ we choose a vertex $h \in X\big(H_g\big)$ and obtain a nontrivial convex 2-cover

$$\mathcal{P}_2\big((G;\{g_1, g_2, \ldots, g_k\}) \odot H\big) = \Big\{\{g\} \cup X\big(H_g\big),$$

$$\{h\} \cup X(G) \cup \bigcup\nolimits_{g' \in \{g_1, g_2, \ldots, g_k\}, g' \neq g} X\big(H_{g'}\big)\Big\}.$$

The theorem is proved. □

Theorem 6.14. *Let G and H be connected graphs of order n and m respectively. Then the following statements hold:*

1) $\varphi_{cn}^{max}(G \odot H) \geq m - \gamma(H)$ if $n = 1$ and $m \geq 3$;

2) $\varphi_{cn}^{max}(G \odot H) = nm$ if $n \geq 2$ and $m \geq 1$.

Proof. 1) Suppose that $n = 1$ and $m \geq 3$. Here, in fact, $G \odot H = K_1 + H$. Then by Theorem 6.6, we get $\varphi_{cn}^{max}(G \odot H) \geq m - \gamma(H)$.

2) Suppose that $n \geq 2$ and $m \geq 1$. Let us consider two cases:

a) Assume that $m = 1$. Note that by covering each vertex $h \in \bigcup_{1 \leq i \leq n} X(H_{x_i})$, where $X(G) = \{x_1, x_2, \ldots, x_n\}$, with nontrivial and convex sets, we simultaneously cover all vertices of $X(G)$. Hence, $\varphi_{cn}^{max}(G \odot H) \leq nm$ (here, $m = 1$). We define a family of nontrivial

convex sets $\mathcal{P}(G \odot H) = \emptyset$ that will cover graph $G \odot H$. We add the set of vertices $\{h, x_i, y\}$ to $\mathcal{P}(G \odot H)$ for each $x_i \in X(G)$, where $h \in X(H_{x_i})$ and $y \in X(G),\ y \sim x_i$. It is clear that $|\mathcal{P}(G \odot H)| = nm$ (here, also, $m = 1$). Thus, if $n \geq 2$ and $m = 1$, then $\varphi_{cn}^{max}(G \odot H) = nm$.

b) Assume $m \geq 2$. We will show that there exists a maximum nontrivial convex cover of $G \odot H$ such that all vertices of $X(G)$ are non-resident in this cover. Let $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$ be a maximum nontrivial convex cover of $G \odot H$. Let x be a vertex of $X(G)$ that is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$, $x \in S, S \in \mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$.

If $|S \cap X(G)| \geq 2$, then by removing from $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$ all sets containing vertices of $\{x\} \cup X(H_x)$, we obtain a family of sets $\mathcal{P}(G \odot H)$. We add the set of vertices $\{y\} \cup (S \backslash X(H_x))$ to $\mathcal{P}(G \odot H)$ for each $y \in X(H_x)$. In this way, we have removed no more than $m - 1$ sets and have added m nontrivial convex sets such that each of them has at least one resident vertex in $\mathcal{P}(G \odot H)$. Thus, $|\mathcal{P}(G \odot H)| > |\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)|$. This contradiction yields that $|S \cap X(G)| = 1$.

We choose an arbitrary vertex $y \in \Gamma(x) \cap X(G),\ y \in C$, where $C \in \mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$. We define a nontrivial convex cover $\mathcal{P}(G \odot H)$ consisting of sets from $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$ except for sets which contain vertices of $\{x\} \cup X(H_x)$. If y is a resident vertex in $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$ and C does not contain another resident vertex in $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$, then we also remove set C from $\mathcal{P}(G \odot H)$. Finally, for each vertex $z \in X(H_x)$, we add the set $\{x, y, z\}$ to $\mathcal{P}(G \odot H)$. It can be easily seen that family $\mathcal{P}(G \odot H)$ contains at least the same number of nontrivial and convex sets as $\mathcal{P}_{\varphi_{cn}^{max}}(G \odot H)$, and all vertices of $X(G)$ are non-resident in $\mathcal{P}(G \odot H)$. It can be concluded that any nontrivial convex cover of $G \odot H$ has at most nm sets.

Now, we define a family of sets $\mathcal{P}(G \odot H)$ in which all vertices of $X(G)$ are non-resident and all vertices of each $X(H_{x_i}),\ 1 \leq i \leq n$, are resident in $G \odot H$ and belong to different sets of $\mathcal{P}(G \odot H)$. The family $\mathcal{P}(G \odot H)$ consists of sets $\{y, x_i, h_j\}$ for all vertices $x_i \in X(G), 1 \leq i \leq n$, and for all vertices $h_j \in X(H_{x_i}),\ 1 \leq j \leq m$, where $y \in X(G),\ y \sim x_i$. Since $|\mathcal{P}(G \odot H)| = nm$, this family is a maximum nontrivial convex cover of $G \odot H$, and furthermore $\varphi_{cn}^{max}(G \odot H) = nm$.

6.3 Cartesian product

The *cartesian product* of two graphs G and H is the graph $G \times H$ on vertex set $X(G) \times X(H)$ in which vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either $g_1 = g_2$ and $h_1 h_2 \in U(H)$ or $h_1 = h_2$ and $g_1 g_2 \in U(G)$.

The map $p_G: X(G) \times X(H) \to X(G), p_G((g,h)) = g$, is the *projection* onto G and $p_H: X(G) \times X(H) \to X(H)$, $p_H((g,h)) = h$, the projection onto H.

Theorem 6.15. [21] *Let G and H be two connected graphs. Then a subset C of $X(G \times H)$ is convex in $G \times H$ if and only if $p_G(C)$ is convex in G, $p_H(C)$ is convex in H, and $C = p_G(C) \times p_H(C)$.*

Theorem 6.16. *Let G be a connected graph of order n and K_m a complete graph of order m such that $n + m \geq 3$. Then, the following statements hold:*

1) $\varphi_c^{min}(G \times K_m) = \varphi_c^{min}(G)$ if $m = 1$;

2) $\varphi_{cn}^{min}(G \times K_m) = \varphi_{cn}^{min}(G)$ if $m = 1$ and $n \geq 4$;

3) $\varphi_c^{min}(G \times K_m) = 2$ if $m \geq 2$;

4) $\varphi_{cn}^{min}(G \times K_m) = 2$ if $m \geq 2$ and $n \geq 3$, or $m \geq 3$ and $n \geq 2$, or $m \geq 4$.

Proof. 1) Suppose $m = 1$. Here, $G = G \times K_1$. Since $n + m \geq 3$, it is evident that

$$\varphi_c^{min}(G \times K_m) = \varphi_c^{min}(G).$$

Further, assume that $n \geq 4$. In this case, graph $G \times K_1$ has a nontrivial convex cover if and only if G has a nontrivial convex cover. Consequently, we have

$$\varphi_{cn}^{min}(G \times K_m) = \varphi_{cn}^{min}(G).$$

Thus, statement 2) also holds.

3) Suppose $m \geq 2$. We choose two different vertices $k_1, k_2 \in X(K_m)$ and obtain two sets:

$$C_1 = \{(g,k) : g \in X(G), k \in X(K_m) \backslash \{k_1\}\} \text{ and}$$

$$C_2 = \{(g,k) : g \in X(G), k \in X(K_m) \backslash \{k_2\}\}.$$

Since K_m is a complete graph, both sets C_1 and C_2 satisfy the conditions of Theorem 6.15. Furthermore, sets C_1 and C_2 form a convex 2-cover of $G \times K_m$ and we obtain

$$\varphi_c^{min}(G \times K_m) = 2.$$

If $n \geq 3$, then C_1 and C_2 form a nontrivial convex 2-cover of $G \times K_m$ and further $\varphi_{cn}^{min}(G \times K_m) = 2$. Similarly, if $m \geq 3$ and $n \geq 2$, or if $m \geq 4$, then we also get

$$\varphi_{cn}^{min}(G \times K_m) = 2.$$

The theorem is proved □

Theorem 6.17. *Let G and H be two connected non-complete graphs and $P(G) = \bigcup_{S \in \mathcal{P}_{\varphi_c^{min}(G \times H)}} \{p_G(S)\}$. Then either*

a) $|P(G)| = 1$, *or*

b) $|P(G) \backslash \{X(G)\}| \geq 2$.

Proof. Let $\mathcal{P}_{\varphi_c^{min}}(G \times H)$ be a minimum convex cover of graph $G \times H$. If $|P(G)| = 1$, then $P(G) = \{X(G)\}$. Also, it is clear that the inequality $|P(G) \backslash \{X(G)\}| \geq 2$ may occur.

Now, assume that $|P(G) \backslash \{X(G)\}| = 1$. Further, for $S \in P(G) \backslash \{X(G)\}$ there is a set $S' \in \mathcal{P}_{\varphi_c^{min}}(G \times H)$ such that $p_G(S') = S$. If $X(G) \notin P(G)$, then we obtain a contradiction because $X(G) \backslash S \neq \emptyset$, which means that graph $G \times H$ is not covered by convex sets. Suppose further that $X(G) \in P(G)$. From the definition of convex cover, we know that every set of the family $\mathcal{P}_{\varphi_c^{min}}(G \times H)$ has a resident vertex. Hence, there exists a vertex $h \in X(H)$ for which there is a vertex (g, h) of $G \times H$ that belongs to S' and does not belong to the set $S'' \in \mathcal{P}_{\varphi_c^{min}}(G \times H)$, where $p_G(S'') = X(G)$. By Theorem 6.15, for h and vertex $g \in X(G) \backslash S$, remain uncovered vertices (g, h) in $G \times H$ and, in consequence, we get a contradiction. Hence, the theorem is proved. □

Corollary 6.4. *Let G and H be two connected non-complete graphs and $P(H) = \bigcup_{S \in \mathcal{P}_{\varphi_c^{min}(G \times H)}} \{p_H(S)\}$. Then either*

a) $|P(H)| = 1$, *or*

b) $|P(H) \backslash \{X(H)\}| \geq 2$.

Theorem 6.18. *Let G and H be two connected non-complete graphs. Then the following equalities hold:*

$$\varphi_c^{min}(G \times H) = \varphi_{cn}^{min}(G \times H) = min\{\varphi_c^{min}(G), \varphi_c^{min}(H)\}.$$

Proof. First, we have to mention that $|X(G)| \geq 3$ and $|X(H)| \geq 3$. By Proposition 6.2, there is a minimum convex cover $\mathcal{P}_{\varphi_c^{min}}(G)$ of G such that every set of $\mathcal{P}_{\varphi_c^{min}}(G)$ has at least two elements. Further, by Theorem 6.15, we get a nontrivial convex cover $\mathcal{P}(G \times H)$ of $G \times H$ that consists of sets $C_i = \{(g,h) : g \in S_i, h \in X(H)\}, S_i \in \mathcal{P}_{\varphi_c^{min}}(G), 1 \leq i \leq \varphi_c^{min}(G)$. Note that $|\mathcal{P}(G \times H)| = \varphi_c^{min}(G)$. This implies $\varphi_{cn}^{min}(G \times H) \leq \varphi_c^{min}(G)$. For the same reason, $\varphi_{cn}^{min}(G \times H) \leq \varphi_c^{min}(H)$. Hence, we have

$$\varphi_c^{min}(G \times H) \leq \varphi_{cn}^{min}(G \times H) \leq min\{\varphi_c^{min}(G), \varphi_c^{min}(H)\}.$$

Let $\mathcal{P}_{\varphi_c^{min}}(G \times H)$ be a minimum convex cover of $G \times H$. Using Theorem 6.15, we get two families of sets:

$$P(G) = \bigcup\nolimits_{S \in \mathcal{P}_{\varphi_c^{min}}(G \times H)} \{p_G(S)\},$$

$$P(H) = \bigcup\nolimits_{S \in \mathcal{P}_{\varphi_c^{min}}(G \times H)} \{p_H(S)\}.$$

But it is clear that the equalities $|P(G)| = 1$ and $|P(H)| = 1$ do not hold at the same time. So, by Theorem 6.17 and Corollary 6.4, we consider three cases.

Suppose that $|P(G)| = 1$. In this case, $|P(H)\backslash\{X(H)\}| \geq 2$. Consequently, for $\mathcal{P}(P(H))$, we get $|\mathcal{P}(P(H))| \leq \varphi_c^{min}(G \times H)$ and also $\varphi_c^{min}(H) \leq \varphi_c^{min}(G \times H)$. Now, suppose that $|P(H)| = 1$. As above, we have $|\mathcal{P}(P(G))| \leq \varphi_c^{min}(G \times H)$ and $\varphi_c^{min}(G) \leq \varphi_c^{min}(G \times H)$. And by the same way, if $|P(G)\backslash\{X(G)\}| \geq 2$ and $|P(H)\backslash\{X(H)\}| \geq 2$, we get $\varphi_c^{min}(G) \leq \varphi_c^{min}(G \times H)$ and $\varphi_c^{min}(H) \leq \varphi_c^{min}(G \times H)$. Combining these three cases, we have

$$min\{\varphi_c^{min}(G), \varphi_c^{min}(H)\} \leq \varphi_c^{min}(G \times H).$$

Finally, we obtain

$$\varphi_c^{min}(G \times H) = \varphi_{cn}^{min}(G \times H) = min\{\varphi_c^{min}(G), \varphi_c^{min}(H)\}.$$

The theorem is proved. □

6.4 Lexicographic product

The *lexicographic product* of two graphs G and H, denoted $G \circ H$, is the graph on vertex set $X(G) \times X(H)$ in which vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either $g_1 g_2 \in U(G)$ or $g_1 = g_2$ and $h_1 h_2 \in U(H)$. The graph $G \circ H$ is called *nontrivial* if both graphs G and H have at least two vertices.

Theorem 6.19. [24] *Let C be a proper subset of a nontrivial connected lexicographic product $G \circ H$. If C induces a non-complete subgraph of $G \circ H$, then C is convex if and only if the following conditions hold:*

a) $p_G(C)$ is convex in G;

b) $\{g\} \times X(H) \subseteq C$ for any non-simplicial vertex $g \in p_G(C)$;

c) H is complete.

The following corollary derives directly from Theorem 6.19.

Corollary 6.5. *Let C be a proper subset of a nontrivial connected lexicographic product $G \circ H$, where H is non-complete. Then C is convex if and only if it induces a complete subgraph of $G \circ H$ and the following conditions hold:*

a) $p_G(C)$ induces a complete subgraph of G;

b) for every $g \in p_G(C)$, set $p_H(C^g)$ induces a complete subgraph of H, where $C^g = \{(g, h) \in C : for\ any\ h \in H\}$.

Theorem 6.20. *Let G be a connected graph of order n and K_m a complete graph of order m such that $n + m \geq 3$. Then the following statements hold:*

1) $\varphi_c^{min}(G \circ K_m) = \varphi_c^{min}(K_m \circ G) = 2$ if G is complete;

2) $\varphi_{cn}^{min}(G \circ K_m) = \varphi_{cn}^{min}(K_m \circ G) = 2$ if G is complete, $n = 2$ and $m = 2$, or $n + m \geq 5$;

3) $\varphi_c^{min}(G \circ K_m) = \varphi_c^{min}(K_m \circ G) = \varphi_c^{min}(G)$ if G is non-complete and $m = 1$;

4) $\varphi_{cn}^{min}(G \circ K_m) = \varphi_{cn}^{min}(K_m \circ G) = \varphi_{cn}^{min}(G)$ if G is non-complete and $n \geq 4$, $m = 1$;

5) $\varphi_c^{min}(G \circ K_m) = \varphi_{cn}^{min}(G \circ K_m) = 2$ *if* G *is non-complete, has a simplicial vertex and* $m \geq 2$*;*

6) $\varphi_c^{min}(G \circ K_m) = \varphi_{cn}^{min}(G \circ K_m) = \varphi_c^{min}(G)$ *if* G *is non-complete, has no simplicial vertices and* $m \geq 2$*;*

7) $\varphi_c^{min}(K_m \circ G) = \varphi_{cn}^{min}(K_m \circ G) = \theta(G)$ *if* G *is non-complete and* $m \geq 2$.

Proof. 1) Suppose that G is complete. Then, it is evident that the obtained graph is complete, and we get

$$\varphi_c^{min}(G \circ K_m) = \varphi_c^{min}(K_m \circ G) = 2.$$

In addition, suppose that $n + m \geq 5$, or $n = 2$ and $m = 2$. The obtained complete graph with at least 4 vertices has a nontrivial convex 2-cover. So, $\varphi_{cn}^{min}(G \circ K_m) = \varphi_{cn}^{min}(K_m \circ G) = 2$ and further statement 2) also holds.

3) Suppose G is non-complete. If $m = 1$, then graphs $G \circ K_m$ and $K_m \circ G$ are equal to G and, further, we have

$$\varphi_c^{min}(G \circ K_m) = \varphi_c^{min}(K_m \circ G) = \varphi_c^{min}(G).$$

In the same way, with condition $n \geq 4$, statement 4) holds. In other words, $\varphi_{cn}^{min}(G \circ K_m) = \varphi_{cn}^{min}(K_m \circ G) = \varphi_{cn}^{min}(G)$. Assume that $m \geq 2$. If G has a simplicial vertex g', then we choose two different vertices $k_1, k_2 \in X(K_m)$ and obtain two sets:

$$C_1 = \big((X(G)\backslash\{g'\}) \times X(K_m)\big) \cup \big\{(g', k) : k \in X(K_m)\backslash\{k_1\}\big\},$$

$$C_2 = \big((X(G)\backslash\{g'\}) \times X(K_m)\big) \cup \big\{(g', k) : k \in X(K_m)\backslash\{k_2\}\big\}.$$

Sets C_1 and C_2 satisfy the conditions of Theorem 6.19 and these two sets form a nontrivial convex 2-cover of $G \circ K_m$. So, we have

$$\varphi_c^{min}(G \circ K_m) = \varphi_{cn}^{min}(G \circ K_m) = 2.$$

Thus, statement 5) is satisfied.

Now assume that G has no simplicial vertices. We know from Theorem 6.19 that for every convex set C of $G \circ K_m$ the projection $p_G(C)$ must be convex in G. Let $\mathcal{P}_{\varphi_c^{min}}(G \circ K_m)$ be a minimum convex cover of $G \circ K_m$. We get the family

$$P(G) = \bigcup\nolimits_{S \in \mathcal{P}_{\varphi_c^{min}}(G \circ K_m)} \{p_G(S)\}.$$

Since the non-complete graph G has no simplicial vertices, it follows that $P(G)$ has no set $X(G)$. Obviously, for convex cover $\mathcal{P}(P(G))$, $|\mathcal{P}(P(G))| \leq \varphi_c^{min}(G \circ H)$ and further $\varphi_c^{min}(G) \leq \varphi_c^{min}(G \circ K_m)$.

Let $\mathcal{P}_{\varphi_c^{min}}(G)$ be a minimum convex cover of graph G, sets $S_i = C_i \times X(K_m)$ form a convex cover of $G \circ K_m$, where $C_i \in \mathcal{P}_{\varphi_c^{min}}(G)$, $1 \leq i \leq \varphi_c^{min}(G)$, and further $\varphi_c^{min}(G \circ K_m) \leq \varphi_c^{min}(G)$. As a result, we have $\varphi_c^{min}(G \circ K_m) = \varphi_c^{min}(G)$. Taking into account Proposition 6.2 the following equalities are true:

$$\varphi_c^{min}(G \circ K_m) = \varphi_{cn}^{min}(G \circ K_m) = \varphi_c^{min}(G).$$

So, statement 6) also holds.

It follows from Corollary 6.5 that every proper convex subset of the product $K_m \circ G$ is a clique and, further, by Propositions 6.2 and 6.3 we have

$$\varphi_c^{min}(K_m \circ G) = \varphi_{cn}^{min}(K_m \circ G) = \theta(G).$$

Further, statement 7) is true. □

Theorem 6.21. *Let G and H be two connected non-complete graphs. Then the following equalities hold:*

$$\varphi_c^{min}(G \circ H) = \varphi_{cn}^{min}(G \circ H) = \theta(G \circ H) = \theta(G)\theta(H).$$

Proof. We know from Corollary 6.5 that every convex set of $G \circ H$ is a clique. Further, we have $\varphi_c^{min}(G \circ H) = \theta(G \circ H)$. Moreover, it can be easily checked that $\theta(G \circ H) = \theta(G)\theta(H)$. Taking into account Propositions 6.2 and 6.3, we get $\varphi_c^{min}(G \circ H) = \varphi_{cn}^{min}(G \circ H)$ and

$$\varphi_c^{min}(G \circ H) = \varphi_{cn}^{min}(G \circ H) = \theta(G \circ H) = \theta(G)\theta(H).$$

The theorem is proved. □

Chapter 7
Binary Linear Programming Approach

We assume that the reader is familiar with basic linear programming techniques. A comprehensive treatment of this subject can be found, for example, in [25, 26]. Note that binary linear programming (BLP) is like linear programming, with the additional constraint that all variables must take on the values 0 or 1.

Similarly to section 3.4, the problems of determining the numbers $\varphi_c^{min}(G)$, $\theta_c^{min}(G)$, $\varphi_{cn}^{min}(G)$, $\theta_{cn}^{min}(G)$, $\varphi_{cn}^{max}(G)$ and $\theta_{cn}^{max}(G)$ are denote by MinCC, MinCP, MinNCC, MinNCP, MaxNCC, and MaxNCP, respectively.

Before proceeding to BLP models for solving these problems, we have to recall the properties that should be satisfied by a family of sets $\mathcal{P}(G)$ that forms a convex cover of a graph $G = (X, U)$. These properties are the following three:

a) every set of $\mathcal{P}(G)$ is convex in G;

b) $X = \bigcup_{S \in \mathcal{P}(G)} S$;

c) $S \not\subseteq \bigcup_{\substack{C \in \mathcal{P}(G) \\ C \neq S}} C$ for every set $S \in \mathcal{P}(G)$.

If, in addition to properties a), b) and c), a family $\mathcal{P}(G)$ satisfies the property d), set out below, then $\mathcal{P}(G)$ is a nontrivial convex cover of G.

d) each set $S \in \mathcal{P}(G)$ is nontrivial, i.e., $2 \leq |S| \leq |X| - 1$;

e) $S \cap C = \emptyset$ for any two sets $S, C \in \mathcal{P}(G)$, $S \neq C$.

Finally, if a convex cover $\mathcal{P}(G)$ satisfies the property e), then $\mathcal{P}(G)$ is a convex partition of G.

We consider a graph G with vertex set $X = \{x_1, x_n, \dots, x_n\}$. We define n subsets S_1, S_2, ... , S_n of X, and two types of binary variables. For each vertex $x_j \in X$ there are variables $y_{ij} \in \{0,1\}$, indicating whether x_j belongs or not to S_i. Also, there are variables $z_{lij} \in \{0,1\}$ which enforce the convexity constraints of S_l. We define

$$y_{ij} = \begin{cases} 1, & x_j \in S_i \\ 0, & x_j \notin S_i \end{cases}$$

and

$$z_{lij} = \begin{cases} 1, & \{x_i, x_j\} \subseteq S_l \\ 0, & \{x_i, x_j\} \not\subseteq S_l. \end{cases}$$

We denote by Y the set of all variables y_{ij} and by Z the set of all variables z_{lij}. The BLP model (BLPM) that corresponds to MinCP problem is presented below.

$$\text{minimize} \quad \sum_{i=1}^{n} y_{ii} \tag{1}$$

$$\text{subject to} \quad \sum_{i=1}^{n} y_{ii} \geq 2 \tag{2}$$

$$\sum_{i=1}^{n} y_{ij} = 1, \qquad j = \overline{1,n} \tag{3}$$

$$\sum_{x_k \in \langle x_i, x_j \rangle} y_{lk} - |\langle x_i, x_j \rangle| z_{lij} \geq 0, \qquad l,i,j = \overline{1,n} \tag{4}$$

$$y_{li} + y_{lj} - z_{lij} \leq 1, \qquad l,i,j = \overline{1,n} \tag{5}$$

$$z_{lij} - y_{li} \leq 0, \qquad l,i,j = \overline{1,n} \tag{6}$$

$$z_{lij} - y_{lj} \leq 0, \qquad l,i,j = \overline{1,n} \tag{7}$$

$$y_{ij} - y_{ii} \leq 0, \qquad i,j = \overline{1,n} \tag{8}$$

$$y_{ij}, z_{lij} \in \{0,1\}, \qquad l,i,j = \overline{1,n} \tag{9}$$

Theorem 7.1. *Each feasible solution of BLPM with objective function (1) and restrictions (2)–(9) corresponds to a feasible solution of MinCP problem and vice versa.*

Proof. Let (Y, Z) be a feasible solution of the BLPM. Each entry in (Y, Z) is zero or one because of the integrality condition (9). Constraints (8) ensure that if there is a vertex x_j belonging to S_i, then $x_i \in S_i$. Restrictions (3) require that each $x_i \in X$ belongs to exactly one set of S_1, S_2, ... , S_n.

Considering these constraints, we define $\mathcal{P} = \{S_i : S_i \neq \emptyset, 1 \leq i \leq n\}$ that partitions graph G. By restriction (2), $\mathcal{P}$ consists of at least two sets. It follows that $\mathcal{P}$ satisfies properties b), c), and e). Moreover, constraints (4) – (7) yield that if both x_i and x_j are in the same set S_l then $\langle x_i, x_j \rangle \subseteq S_l$, thus fulfilling the convexity property a) for each set of $\mathcal{P}$. This, therefore, means that $\mathcal{P}$ is an admissible convex partition of G.

Let $\mathcal{P}(G)$ be a convex partition of G such that $|\mathcal{P}(G)| \geq 2$, in other words, it is a feasible solution of MinCP problem. For each set $S \in \mathcal{P}(G)$, we choose one resident vertex $x_i \in S$ in $\mathcal{P}(G)$ and set y_{ij} to one if $x_j \in S$, otherwise, set it to zero. In the same manner, we set z_{lij} to one if $y_{li} = 1$ and $y_{lj} = 1$, otherwise set it to zero. In light of all the above arguments, we can conclude that (Y, Z) does not violate any constraints. For this reason, (Y, Z) is a feasible solution for the BLPM. □

Corollary 7.1. *An optimal solution of BLPM with objective function (1) and restrictions (2) – (9) corresponds to an optimal solution of MinCP problem.*

Now, we present the BLP model for the MaxNCC problem, which is constructed similarly to the above model.

$$\text{maximize} \quad \sum_{i=1}^{n} y_{ii} \tag{10}$$

$$\text{subject to} \quad \sum_{i=1}^{n} y_{ii} \geq 2 \tag{11}$$

$$\sum_{j=1, j\neq i}^{n} y_{ij} - 2y_{ii} \geq 0, \qquad i = \overline{1,n} \tag{12}$$

$$\sum_{i=1, i\neq j}^{n} y_{ij} + My_{jj} \geq 1, \qquad j = \overline{1,n} \tag{13}$$

$$\sum_{i=1, i\neq j}^{n} y_{ij} + My_{jj} \leq M, \qquad j = \overline{1,n} \tag{14}$$

$$\sum_{x_k \in \langle x_i, x_j \rangle} y_{lk} - |\langle x_i, x_j \rangle| z_{lij} \geq 0, \qquad l, i, j = \overline{1,n} \tag{15}$$

$$y_{li} + y_{lj} - z_{lij} \leq 1, \qquad l, i, j = \overline{1,n} \tag{16}$$

$$z_{lij} - y_{li} \leq 0, \qquad l, i, j = \overline{1,n} \tag{17}$$

$$z_{lij} - y_{lj} \leq 0, \qquad l, i, j = \overline{1,n} \tag{18}$$

$$y_{ij} - y_{ii} \leq 0, \qquad i, j = \overline{1,n} \tag{19}$$

$$y_{ij}, z_{lij} \in \{0,1\}, \qquad l, i, j = \overline{1,n} \tag{20}$$

Theorem 7.2. *Each feasible solution of BLPM with objective function (10) and restrictions (11)–(20) corresponds to a feasible solution of MaxNCC problem and vice versa.*

Proof. Let (Y, Z) be a feasible solution of the BLPM. Restriction (20) enforce the binary nature of the variables y_{ij} and z_{lij}. By constraints (19), each nonempty set S_i contains vertex x_i. Moreover, constraints (13) and (14) mean that for each nonempty set S_j the vertex x_j is not contained in $\bigcup_{1\le i\le n, i\ne j} S_i$, and there is no vertex x_i such that $x_i \notin \bigcup_{1\le j\le n} S_j$. Considering these constraints, we define $\mathcal{P} = \{S_i : S_i \ne \emptyset, 1 \le i \le n\}$ that covers graph G. By restriction (11), $|\mathcal{P}| \ge 2$. Constraints (12) yield that every set $S_i \in \mathcal{P}$ has at least three elements. This ensures that every set of $\mathcal{P}$ is nontrivial. It is clear that $\mathcal{P}$ satisfies properties b), c), and d). The remaining constraints (15) – (18) imply that each set of $\mathcal{P}$ is convex, thus property a) also holds. Therefore, for a feasible solution of the BLPM, there is a feasible solution of MaxNCC problem.

As in the proof of Theorem 7.1, it can easily be verified that for a feasible solution of MaxNCC problem there exists a feasible solution of the BLPM.

The theorem is proved. □

Corollary 7.2. *An optimal solution of BLPM with objective function (10) and restrictions (11)–(20) corresponds to an optimal solution of MaxNCC problem.*

Regarding optimization problems MinCC, MinNCC, MinNCP and MaxNCP, we easily get the following corollaries:

Corollary 7.3. *An optimal solution of BLPM with objective function (1) and restrictions (11), (13)–(20) corresponds to an optimal solution of MinCC problem.*

Corollary 7.4. *An optimal solution of BLPM with objective function (1) and restrictions (11)–(20) corresponds to an optimal solution of MinNCC problem.*

Corollary 7.5. *An optimal solution of BLPM with objective function (1) and restrictions (2)–(9), (12) corresponds to an optimal solution of MinNCP problem.*

Corollary 7.6. *An optimal solution of BLPM with objective function (10) and restrictions (2)–(9), (12) corresponds to an optimal solution of MaxNCP problem.*

The above shown is a description of BLP models designed to solve optimization problems of convex covering problem of graphs. We should mention that similar BLP models can be formulated for general convex p-cover problems of graphs, where $p \geq 2$ is a fixed integer.

For example, by replacing objective function (1) with a constant and restriction (2) with

$$\sum_{i=1}^{n} y_{ii} = p \tag{21}$$

we obtain the BLPM for convex p-partition problem with constant objective function and restrictions (3) – (9), (21).

In view of the proofs of Theorems 7.1 and 7.2, the correctness of the following corollaries follows:

Corollary 7.7. *An optimal solution of BLPM with constant objective function and restrictions (3)–(9), (21) corresponds to a feasible solution of convex p-partition problem.*

Corollary 7.8. *An optimal solution of BLPM with constant objective function and restrictions (3)–(9), (12), (21) corresponds to a feasible solution of nontrivial convex p-partition problem.*

Corollary 7.9. *An optimal solution of BLPM with constant objective function and restrictions (13)–(20), (21) corresponds to a feasible solution of convex p-cover problem.*

Corollary 7.10. *An optimal solution of BLPM with constant objective function and restrictions (12)–(20), (21) corresponds to a feasible solution of nontrivial convex p-cover.*

And, finally, we have to formulate BLP models for nontrivial convex p-cover and nontrivial convex p-partition problems in the case when $p \geq 2$ is not fixed. Similarly to section 3.3, we refer to these problems as NCC and NCP, respectively.

Based on the proof of Theorems 7.1 and 7.2, the following corollaries follow:

Corollary 7.11. *An optimal solution of BLPM with constant objective function and restrictions (11)–(20) corresponds to a feasible solution of NCC problem.*

Corollary 7.12. *An optimal solution of BLPM with constant objective function and restrictions (2)–(9), (12) corresponds to a feasible solution of NCP problem.*

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Symbol Index

$|S|$ Cardinality of set S, 7
$\langle x, y\rangle$ Metric segment between vertices x and y, 9
$\alpha(G)$ Independence number of G, 8
$\beta(G)$ Edge cover number of G, 8
$\gamma(G)$ Domination number of G, 8
$\Gamma(x)$ Neighborhood of x, 7
$\theta(G)$ Clique cover number of G, 77
$\theta_c^{min}(G)$ Minimum convex partition number of G, 10
$\theta_{cn}^{max}(G)$ Maximum nontrivial convex partition number of G, 10
$\theta_{cn}^{min}(G)$ Minimum nontrivial convex partition number of G, 10
$\varphi_c^{min}(G)$ Minimum convex cover number of G, 10
$\varphi_{cn}^{max}(G)$ Maximum nontrivial convex cover number of G, 10
$\varphi_{cn}^{min}(G)$ Minimum nontrivial convex cover number of G, 10
$\mathcal{A}$ Class of graphs, 67
$\mathcal{B}$ Class of graphs, 67
C_n Cycle graph of order n, 8
C_n^k Power of cycle of order n, 53
$d(x, y)$ Distance between vertices x and y, 9
$d - conv(S)$ Convex hull of S, 9
$deg(x)$ Degree of vertex x, 7
$diam(G)$ Diameter of G, 9
$\mathcal{E}(G)$ Family of subtrees obtained after removing $\mathcal{S}(G)$ from a tree G, 73
$\mathcal{F}$ Class of graphs, 42
$G[S]$ Subgraph of G induced by $S \subseteq X(G)$, 7
$G + H$ Join of graphs G and H, 78
$G \times H$ Cartesian product of graphs G and H (or of sets G and H), 89
$G \circ H$ Lexicographic product of graphs G and H, 92
$G \odot H$ Corona product of graphs G and H, 85
$\mathcal{H}$ Class of graphs, 45

$\mathcal{H}'$ Class of graphs, 45
$\mathcal{H}''$ Class of graphs, 45
$\mathcal{J}$ Class of graphs, 45
K_n Complete graph of order n, 8
$K_{p,q}$ Complete bipartite graph of order $p+q$, 8
$\mathcal{P}(G)$ Convex cover of G, 9
$\mathcal{P}_p(G)$ Convex p-cover of G, 9
$\mathcal{P}_{2,nt}(G)$ Convex $(2,nt)$-cover of G, 38
$\mathcal{P}_{2,t}(G)$ Convex $(2,t)$-cover of G, 38
$\widetilde{\mathcal{P}}_{2,t}(G)$ Family of all convex $(2,t)$-covers of G, 38
$\mathcal{P}_\theta(G)$ Minimum clique cover of G, 77
$\mathcal{P}_{\theta_c^{min}}(G)$ Minimum convex partition of G, 10
$\mathcal{P}_{\theta_{cn}^{max}}(G)$ Maximum nontrivial convex partition of G, 10
$\mathcal{P}_{\theta_{cn}^{min}}(G)$ Minimum nontrivial convex partition of G, 10
$\mathcal{P}_{\varphi_c^{min}}(G)$ Minimum convex cover of G, 10
$\mathcal{P}_{\varphi_{cn}^{max}}(G)$ Maximum nontrivial convex cover of G, 10
$\mathcal{P}_{\varphi_{cn}^{min}}(G)$ Minimum nontrivial convex cover of G, 10
$p_G(W)$ Projection of $W \subseteq X(G) \times X(H)$ into G, 89
$\mathcal{S}(G)$ Family of all nontrivial terminal sets of a tree G, 70
$U(G)$ Edge set of G, 7
$X(G)$ Vertex set of G, 7
$x \sim y$ Adjacency of vertices x and y, 7

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Monograph

Signed for printing 04.11.2021.
Format 60 × 84 1/16.

Printing sheets 6,7. Editorial sheets 4,0.
Order 136/21. Edition 70 copies.

The Editorial Publishing Center of MSU,
60 Al. Mateevici, Chisinau, MD-2009