Proceedings of the International Conference IMCS-60, MSU
October 10–13, 2024, Chisinau, Republic of Moldova

Strongly Polynomial Primal-Dual Algorithms for Concave Cost Combinatorial Optimization Problems*

Thomas L. Magnanti, Dan Stratila

Abstract

We introduce an algorithm design technique for a class of combinatorial optimization problems with concave costs. This technique yields a strongly polynomial primal-dual algorithm for a concave cost problem whenever such an algorithm exists for the fixed-charge counterpart of the problem.

Using our technique, we obtain a 1.61-approximation algorithm for the concave cost facility location problem, an exact algorithm for the concave cost lot-sizing problem, and a 4-approximation algorithm for the joint replenishment problem with general concave individual ordering costs.

Keywords: Primal-dual algorithms, concave costs, facility location, lot-sizing, joint replenishment problems.

1 Concave Cost Facility Location

In the concave cost facility location problem, there are m customers and n facilities. Each customer i has a demand $d_i > 0$, and needs to be connected to a facility to satisfy this demand. Connecting a customer i to a facility j incurs a connection cost $c_{ij}d_i$; we assume that the c_{ij} are nonnegative and satisfy the metric inequality. Each facility j has a nondecreasing concave cost function $\phi_j : \mathbb{R}_+ \to \mathbb{R}_+$, and we assume

*This abstract is based on [8, 6]. A complete bibliography for this abstract can be found in [8, 6].

©2024 by Thomas L. Magnanti, Dan Stratila

without loss of generality that $\phi_j(0) = 0$. Let $x_{ij} = 1$ if customer i is connected to facility j, and $x_{ij} = 0$ otherwise. The cost at facility j is a function of the total demand at j, that is $\phi_j\big(\sum_{i=1}^m d_i x_{ij}\big)$, and the total cost is $\sum_{j=1}^n \phi_j\big(\sum_{i=1}^m d_i x_{ij}\big) + \sum_{i=1}^m \sum_{j=1}^n c_{ij} d_i x_{ij}$. The goal is to minimize the total cost.

Let $[n] = \{1, \ldots, n\}$. The concave cost facility location problem can be written as a mathematical program:

$$\min\Big\{\textstyle\sum_{j=1}^n \phi_j \left(\sum_{i=1}^m d_i x_{ij}\right) + \sum_{i=1}^m \sum_{j=1}^n c_{ij} d_i x_{ij} \;\Big|\; \sum_{j=1}^n x_{ij} = 1, x_{ij} \geq 0, i \in [m], j \in [n]\Big\}. \quad (1)$$

We omit the constraints $x_{ij} \in \{0, 1\}$, which are automatically satisfied at any vertex of the feasible polyhedron. Since the objective is concave, this problem always has a vertex optimal solution.

We interpret each concave function ϕ_j as a piecewise-linear function with an infinite number of pieces. For each $p > 0$, we introduce a tangent $f_{jp} + s_{jp}\xi_j$ to ϕ_j at p, with $s_{jp} = \phi_j'(p)$, and $f_{jp} = \phi_j(p) - p s_{jp}$. Here $\phi_j'(p)$ is the right derivative, which exists as ϕ_j is concave.

We have $\phi_j(\xi_j) = \min\{f_{jp} + s_{jp}\xi_j : p > 0\}$ for $\xi_j > 0$. Problem (1) can be written as a classical facility location problem with m customers and an infinite number of facilities. (For background on the classical facility location problem, see [8, 6].) Each tangent p to cost function ϕ_j of facility j in problem (1) corresponds to a facility $\{j, p\}$ in the resulting problem. The resulting infinitely-sized integer program is:

$$\min\Big\{\textstyle\sum_{j=1}^n \sum_{p\geq 0} f_{jp} y_{jp} + \sum_{i=1}^m \sum_{j=1}^n \sum_{p\geq 0} (c_{ij} + s_{jp}) d_i x_{ijp} \;\Big|\; \sum_{j=1}^n \sum_{p\geq 0} x_{ijp} = 1, x_{ijp} \leq y_{jp}, \; y_{jp} \in \{0, 1\}, i \in [m], j \in [n], p \geq 0\Big\}. \quad (2)$$

Consider the 1.61-approximation algorithm for classical facility location of Jain et al. [2], which we will call Algorithm FLPD. Of course, we cannot run this algorithm on problem (2) directly, as it is infinitely-sized. Instead, we obtain technical results that enable us to execute Algorithm FLPD on problem (2) implicitly. More specifically, we devise an algorithm that takes problem (1) as input, runs in strongly

polynomial time, and produces the same assignment of customers to facilities as if Algorithm FLPD were run on problem (2). We call this algorithm ConcaveFLPD.

Theorem 1. *Algorithm* ConcaveFLPD *is a 1.61-approximation algorithm for the concave cost facility location problem, with a running time of* $O(m^3n + mn \log n)$.

Using piecewise-linear approximations, we also obtained a $1.4991+\epsilon$ approximation algorithm for concave cost facility location [4, 8, 5]. The running time is polynomial in $1/\epsilon$ and the size of the input, but is not strongly polynomial. Independently, Romeijn et al. [7] developed strongly polynomial 1.61 and 1.52-approximation algorithms for concave cost facility location, each with a running time of $O(n^4 \log n)$, where n is the higher of the number of customers and facilities.

2 Concave Cost Inventory Problems

We obtain a strongly polynomial exact algorithm for the concave cost lot-sizing problem with a running time of $O(n^2)$, where n is the number of time periods. To develop this algorithm, we use our technique together with the algorithm of Levi et al. [3] for the classical lot-sizing problem. While the running time matches that of the fastest previous algorithm [1], our main goal is to use this algorithm as a stepping stone in the development of our approximation algorithm for the joint replenishment problem (JRP) with concave individual ordering costs.

We obtain a strongly polynomial 4-approximation algorithm for the JRP with concave individual ordering costs. We call this algorithm ConcaveJRPPD. To develop it, we use our technique together with the 2-approximation algorithm of Levi et al. [3] for the classical JRP.

Theorem 2. *Algorithm* ConcaveJRPPD *is a 4-approximation algorithm for the JRP with concave individual ordering costs.*

Acknowledgments. This research was supported in part by the United States Air Force Office of Scientific Research, Amazon.com, and the Singapore-MIT Alliance. This research was also supported in the

framework of the project "011302: Analitical and numerical methods for solving stochastic dynamic decision problems".

References

[1] A. Aggarwal and J. K. Park. *Improved algorithms for economic lot size problems.* Oper. Res., vol. 41, no. 3, pp. 549–571, 1993.

[2] K. Jain, M. Mahdian, E. Markakis, A. Saberi, and V. V. Vazirani. *Greedy facility location algorithms analyzed using dual fitting with factor-revealing LP.* J. ACM, vol. 50, no. 6, pp. 795–824, 2003.

[3] R. Levi, R. O. Roundy, and D. B. Shmoys. *Primal-dual algorithms for deterministic inventory problems.* Math. Oper. Res., vol. 31, no. 2, pp. 267–284, 2006.

[4] T. L. Magnanti and D. Stratila. *Separable concave optimization approximately equals piecewise linear optimization.* In Integer programming and combinatorial optimization, vol. 3064 of *Lecture Notes in Comput. Sci.*, pp. 234–243. Springer, Berlin, 2004.

[5] T. L. Magnanti and D. Stratila. *Separable concave optimization approximately equals piecewise-linear optimization*, 2012. arXiv preprint arXiv:1201.3148.

[6] T. L. Magnanti and D. Stratila. *Strongly polynomial primal-dual algorithms for concave cost combinatorial optimization problems*, 2012. arXiv preprint arXiv:1202.2654.

[7] H. E. Romeijn, T. C. Sharkey, Z.-J. M. Shen, and J. Zhang. *Integrating facility location and production planning decisions.* Networks, vol. 55, no. 2, pp. 78–89, 2010.

[8] D. Stratila. *Combinatorial optimization problems with concave costs.* PhD thesis, Massachusetts Institute of Technology, Operations Research Center, September 2008.

Thomas L. Magnanti[1], Dan Stratila[2]

[1]School of Engineering and Sloan School of Management, Massachusetts Institute of Technology
E–mail: magnanti@mit.edu
ORCID: https://orcid.org/0000-0002-0771-3850

[2]Vladimir Andrunachievici Institute of Mathematics and Computer Science, Moldova State University
E–mail: dstrat32@gmail.com
ORCID: https://orcid.org/0009-0000-0990-7264