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# Recognition of the simple group $\mathrm{Sz}(q)$, for some $q$ by character degree graph and order

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**Abstract.** Let $G$ be a finite group. The character degree graph of $G$, which is denoted by $\Gamma(G)$, is the graph whose vertices are the prime divisors of the irreducible character degrees of $G$ and two vertices $p$ and $p'$ are joined by an edge if $pp'$ divides some irreducible character degree of $G$. In this paper, we prove that the simple group $\mathrm{Sz}(q)$, for some $q$, is uniquely determined by its character degree graph and its order. This implies that $\mathrm{Sz}(q)$ is uniquely determined by the structure of its complex group algebra.

## 1. Introduction

Let $G$ be a finite group, $\mathrm{Irr}(G)$ be the set of irreducible characters of $G$, and denote by $\mathrm{cd}(G)$, the set of irreducible character degrees of $G$. The *character degree graph* of $G$, which is denoted by $\Gamma(G)$, is defined as follows: the vertices of this graph are the prime divisors of the irreducible character degrees of $G$ and two distinct vertices $p$ and $p'$ are joined by an edge if there exists an irreducible character degree of $G$ which is divisible by $pp'$. This graph has been widely studied (see [17, 19]). A finite group $G$ is called a $K_3$-group if $|G|$ has exactly three distinct prime divisors. Recently Chen et al. in [20] and [21] proved that all simple $K_3$-groups and the Mathieu groups are uniquely determined by their orders and one or both of their largest and second largest irreducible character degrees.

Let $p$ be an odd prime number. In [9], Khosravi et al. proved that the simple group $\mathrm{PSL}(2,p)$ is uniquely determined by its order and its largest and second largest irreducible character degrees. Also as a consequence of their result it follows that the simple group $\mathrm{PSL}(2,p)$ is uniquely determined by its character degree graph and its order (also see [12, 18]). In [1], it is proved that the simple group $\mathrm{PSL}(2,q)$ is recognized by character degree

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graph and order. Also in [8], it is proved that the characteristically simple group $A_5 \times A_5$ is recognizable by character degree graph and order. Also in [13] it is proved that each simple group of order less than 6000 is uniquely determined by its character degree graph and its order. In [14] it isproved that if $p$ is an odd prime number and $G$ is a finite group such that $|G| = |\mathrm{PSL}(2, p^2)|$, $p^2 \in \mathrm{cd}(G)$ and there does not exist any $\theta \in \mathrm{Irr}(G)$ such that $2p \mid \theta(1)$, then $G \cong \mathrm{PSL}(2, p^2)$. In [10], Khosravi et al. proved that if $G$ is a finite group such that $|G| = 2|\mathrm{PSL}(2, p^2)|$, $p^2 \in \mathrm{cd(G)}$ and there does not exist any $\theta \in \mathrm{Irr}(G)$ such that $2p \mid \theta(1)$, then $G$ has a unique nonabelian composition factor isomorphic to $\mathrm{PSL}(2, p^2)$. Similar results were proved for some extensions of $\mathrm{PSL}(2, q)$, for some $q$ (see [11]). In [7] it is proved that the projective special linear group $\mathrm{PSL}(2, q)$ is uniquely determined by its group order and its largest irreducible character degree when $q$ is a prime or when $q = 2^a$ for an integer $a \geqslant 2$ such that $2^a - 1$ or $2^a + 1$ is a prime.

In [15] it is proved that $\mathrm{PSL}(2, p^2)$, where $p$ is a prime, is uniquely determined by its order and its character degree graph. In [2], it is proved that some characteristically simple groups are recognizable by their complex group algebras. In [16], we proved that $\mathrm{PSL}(2, p) \times \mathrm{PSL}(2, p)$ is uniquely determined by the structure of its complex group algebra. In [4], it is proved that Suzuki groups $\mathrm{Sz}(q)$, where $q \pm \sqrt{2q} + 1$ is a prime number can be uniquely determined by the order of group and the number of elements with the same order. In this paper, we prove that the simple group $\mathrm{Sz}(q)$, where $q - \sqrt{2q} + 1$ or $q + \sqrt{2q} + 1$ is a prime number is uniquely determined by its character degree graph and its order.

## 2. Preliminary results

**Definition 2.1.** If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of $\theta$ in $G$ is $I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$. If the character $\chi = \sum_{i=1}^{k} e_i \chi_i$, where for each $1 \leq]slanti \leqslant k$, $\chi_i \in \mathrm{Irr}(G)$ and $e_i$ is a natural number, then each $\chi_i$ is called an irreducible constituent of $\chi$.

**Lemma 2.2.** (Itô's Theorem) [6, Theorem 6.15] *Let $A \trianglelefteq G$ be abelian. Then $\chi(1)$ divides $|G : A|$, for all $\chi \in \mathrm{Irr}(G)$.*

**Lemma 2.3.** [6, Theorems 6.2, 6.8, 11.29] *Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr(G)}$. Let $\theta$ be an irreducible constituent of $\chi_N$ and suppose $\theta_1 = \theta, \ldots, \theta_t$ are the distinct conjugates of $\theta$ in $G$. Then $\chi_N = e \sum_{i=1}^{t} \theta_i$, where $e = [\chi_N, \theta]$ and $t = |G : I_G(\theta)|$. Also $\theta(1) \mid \chi(1)$ and $\chi(1)/\theta(1) \mid |G : N|$.*

**Lemma 2.4.** [20, Lemma] *Let $G$ be a nonsolvable group. Then $G$ has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H$ is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

**Lemma 2.5.** (Itô-Michler Theorem) [5] *Let $\rho(G)$ be the set of all prime divisors of the elements of $\mathrm{cd}(G)$. Then $p \notin \rho(G)$ if and only if $G$ has a normal abelian Sylow $p$-subgroup.*

**Lemma 2.6.** [21, Lemma 2] *Let $G$ be a finite solvable group of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where $p_1, p_2, \ldots, p_n$ are distinct primes. If $(kp_n + 1) \nmid p_i^{\alpha_i}$, for each $i \leqslant n-1$ and $k > 0$, then the Sylow $p_n$-subgroup is normal in $G$.*

If $n$ is an integer and $r$ is a prime number, then we write $r^\alpha \| n$, when $r^\alpha \mid n$ but $r^{\alpha+1} \nmid n$. Also if $r$ is a prime number we denote by $\mathrm{Syl}_r(G)$, the set of Sylow $r$-subgroups of $G$ and we denote by $n_r(G)$, the number of elements of $\mathrm{Syl}_r(G)$. If $H$ is a characteristic subgroup of $G$, we write $H$ ch $G$. All other notations are standard and we refer to [3].

## 3. The main results

**Remark 3.1.** [17] Let $G = \mathrm{Sz}(q)$, where $q = 2^{2m+1} \geqslant 8$. Throughout of this section let $a = q + \sqrt{2q} + 1$ and $b = q - \sqrt{2q} + 1$. Then $|G| = q^2(q^2+1)(q-1)$ and $\mathrm{cd}(G) = \{1, q^2, q^2+1, (q-1)a, (q-1)b, 2^m(q-1)\}$. It is obvious that $(a, b) = 1$, $(q^2+1, q-1) = 1$ and $(q^2+1, q) = 1$.

**Lemma 3.2.** *Let $q = 2^{2m+1}$ and $b = q - \sqrt{2q} + 1$, where $m > 1$ is an integer. Then*

1. *$(kb+1) \nmid q^2$, for every $k \in \mathbb{N}$.*
2. *There is not any divisor $d$ of $a = q + \sqrt{2q} + 1$, where $d = kb + 1$, for some $k \in \mathbb{N}$.*
3. *There is not any divisor $d$ of $q - 1$, where $d = kb + 1$, for some $k \in \mathbb{N}$.*

*Proof.* 1. Since $b - 1 \equiv 2^{m+1} \pmod{2^m - 1}$, so $2, 2^2, \ldots, 2^{2m} \in [2, b-1]$. Therefore $\mathrm{ord}_b 2 \notin [2, b-1]$. It is obvious that

$$2^{2m+1} \equiv 2^{m+1} - 1 \pmod{b} \tag{1}$$

Also $2^{m+1} - 1 \leqslant b - 1 = 2^{m+1}(2^m - 1)$. So $2^{2m+1} \in [2, b-1]$. Now by (1), we have $2^{3m} \equiv 2^{2m} - 2^{m-1} \pmod{b}$. Hence $3m < \mathrm{ord}_b 2$. Continuing in this manner we conclude that $2^{3m+1}, 2^{3m+2}, \ldots, 2^{4m+1} \in [2, b-1]$.

Now we claim that $2^{4m+2} \equiv -1(\bmod b)$. Using (1), we have $2^{4m+1} \equiv 2^{3m+1} - 2^{2m}(\bmod b)$. Therefore $2^{4m+2} \equiv 2^{2m+2} - 2^{m+1} - 2^{2m+1} \equiv -1 \,(\bmod b)$. Up to now, we prove that $\mathrm{ord}_b 2 > 4m+2$. On the other hand, $b \mid (q^4-1)$. Hence $\mathrm{ord}_b 2 \mid 4(2m+1)$. Thus $\mathrm{ord}_b 2 = 8m+4$.

If $(kb+1) \mid q^2$, for some $k \geqslant 1$, then $kb+1 = 2^t$ for some $t$ such that $t \leqslant 4m+2$. So $2^t \equiv 1 \pmod{b}$, which is a contradiction. Hence the proof is completed.

2. We know that $q+1 > 3 \cdot 2^{m+1} = 3\sqrt{2q}$. Thus $2b > a$ and so $(kb+1) \nmid a$, for any $k \geqslant 1$.

3. Since $2b > (q-1)$, so $q+3 > 2\sqrt{2q}$. Therefore $kb+1 > q-1$ and so $(kb+1) \nmid (q-1)$. □

**Lemma 3.3.** *Let $q = 2^{2m+1}$ and $a = q + \sqrt{2q} + 1$, where $m \geqslant 1$ is an integer. Then $(ka+1) \nmid q^2$, for every $k \in \mathbb{N}$.*

*Proof.* Similarly to the proof of Lemma 3.2, we can get the result. □

**Lemma 3.4.** *Let $q' = 2^{2\alpha+1}$ and $q = 2^{2\beta+1}$. Then $|\mathrm{Sz}(q')| \mid |\mathrm{Sz}(q)|$ if and only if $q = q'^s$, for some odd $s$.*

*Proof.* First suppose that $|\mathrm{Sz}(q')| \mid |\mathrm{Sz}(q)|$. Let $p_1$ be a primitive prime divisor of $2^{(2\alpha+1)} - 1$, i.e. $\mathrm{ord}^2_{p_1} = 2\alpha+1$. Therefore by Remark 3.1, $p_1 \mid (q-1)$ or $p_1 \mid (q^2+1)$. If $p_1 \mid (q-1)$, then $(2\alpha+1) \mid (2\beta+1)$. Now suppose that $p_1 \mid (q^2+1)$. So $p_1 \mid (2^{4(2\beta+1)} - 1)$. Therefore $(2\alpha+1) \mid (2\beta+1)$. Consequently, $q = q'^s$, for some odd $s$.

If $q = q'^s$, for some odd $s$, then by Remark 3.1, the proof is obvious. □

**Theorem 3.5.** *Let $q = 2^{2m+1}$ and $q - \sqrt{2q} + 1$ or $q + \sqrt{2q} + 1$ be a prime number, where $m \in \mathbb{N}$. Also let $G$ be a finite group of order $|\mathrm{Sz}(q)|$ such that $\Gamma(G) = \Gamma(\mathrm{Sz}(q))$, then $G \cong \mathrm{Sz}(q)$.*

*Proof.* **Step 1.** Let $q - \sqrt{2q} + 1 = p$ or $q + \sqrt{2q} + 1 = p$ be a prime number. Then we prove that $O_p(G) = 1$. For convenience, we give the proof only when $q + \sqrt{2q} + 1 = p$ be a prime number. Let $O_p(G) \neq 1$. By Remark 3.1, $|O_p(G)| = p$. Using Lemma 2.2, $p \mid \chi(1) \mid [G : O_p(G)]$, for all $\chi \in \mathrm{Irr}(G)$, which is a contradiction. Therefore $O_p(G) = 1$.

**Step 2.** We prove that $G$ is a nonsolvable group. Using Lemmas 2.6, 3.2 and 3.3, the Sylow $p$-subgroup is normal in $G$, which is a contradiction by Step 1. Therefore $G$ is a nonsolvable group.

**Step 3.** We prove that $G \cong \mathrm{Sz}(q)$. By Step 2 and Lemma 2.4, $G$ has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H$ is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. Since $3 \nmid |\mathrm{Sz}(2^{2m+1})|$, for $m \geqslant 1$, so $K/H \cong \mathrm{Sz}(q') \times \mathrm{Sz}(q') \times \ldots \times \mathrm{Sz}(q')$, the direct product of $n$-copies of $\mathrm{Sz}(q')$, such that $q' \mid q$. If $n > 1$, then since $p \| |G|$, so $p \nmid |K/H|$. Therefore $p \mid |G/K|$ or $p \mid |H|$.

Now let $p \mid |G/K|$. Note that $\mathrm{Out}(K/H) \cong \mathrm{Out}(\mathrm{Sz}(q')) \wr S_n$. Since $|\mathrm{Out}(K/H)| \mid (2m+1)^n \times n!$ and $p > 2m+1$ is prime, so $p \nmid (2m+1)^n$. Hence $p \mid |S_n|$, i.e. $p < n$. We know that $29120 = |\mathrm{Sz}(8)| \leqslant |\mathrm{Sz}(q')|$. Consequently, $29120^p \leqslant |K/H| < |G|$, which is a contradiction since $p \| |G|$.

Therefore $p \mid |H|$. Since $H \trianglelefteq G$, so $p \mid \theta(1)$, for some irreducible character $\theta$ of $H$ by Lemma 2.3. Now we consider the following cases:

**(a)** Let $H$ be solvable. We denote by $T$, the $p$-Sylow subgroup of $H$. Using Lemmas 2.6, 3.2 and 3.3, $T \trianglelefteq G$, which is a contradiction.

**(b)** Let $H$ be nonsolvable. By Lemma 2.4, $G$ has a normal series $1 \trianglelefteq A \trianglelefteq B \trianglelefteq H$ such that $B/A$ is a direct product of isomorphic nonabelian simple groups and $|H/B| \mid |\mathrm{Out}(B/A)|$. Similarly to the above, we can prove that $p \nmid |H/B|$ and $p \nmid |B/A|$. So similarly $p \mid |A|$ and $A$ is solvable, which is a contradiction by Lemmas 2.6, 3.2 and 3.3.

Consequently, $p \mid |K/H|$ and $n = 1$. Thus $K/H \cong \mathrm{Sz}(q')$, where $q' \mid q$, by Lemma 3.4. Now since $p \| |G|$ and $p \| |K/H|$, so $|G| = |K/H|$. Therefore $G \cong \mathrm{Sz}(q)$. □

As a consequence we get the following result:

**Corollary 3.6.** *Let $q = 2^{2m+1} \geqslant 8$ and $H = \mathrm{Sz}(q)$. Also let $q - \sqrt{2q} + 1$ or $q + \sqrt{2q} + 1$ be a prime number. If $G$ is a group such that $\mathbb{C}G \cong \mathbb{C}H$, then $G \cong \mathrm{Sz}(q)$. Thus $\mathrm{Sz}(q)$ is uniquely determined by the structure of its complex group algebra.*

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